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Octopus: An Efficient Phase Space Mapping for Light Particles

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I present a generator for relativistic phase space that incorporates much of the effect of typical experimental cuts, and which is suitable for use in Monte Carlo calculations of cross sections for high-energy hadron–hadron or electron–positron scattering experiments. © 1992 Academic Press, Inc.

1. INTRODUCTION

The calculation of cross sections for scattering processes is a basic tool in the analysis of data and backgrounds at both hadron–hadron and electron–positron colliders. The most widely used approach is that of Monte Carlo integration: one generates a set of momenta with a Lorentz-invariant phase space distribution, rejects those sets that fail a set of cuts designed to mimic the experimental cuts (for example, a minimum transverse energy cut in hadron–hadron collisions), and evaluates the relevant matrix element on the remainder, thereby obtaining a numerical estimate of the desired cross section.

The purpose of the present work is to present another algorithm for generating a Lorentz-invariant phase space distribution. It is useful to distinguish two steps in such a process: one typically generates a set of points in a hypercube, $x \in [0, 1]^{d(n)}$, and then maps the hypercube to phase space ($d(n)$ depends on both the number of final-state particles and on the mapping). To calculate a cross section, one integrates over the hypercube a matrix element, multiplied by a weight factor (which in general depends on x). In the simplest approach, one simply generates a set of pseudo-random points distributed uniformly in the hypercube; but more sophisticated adaptive approaches, such as the Vegas algorithm [1], are also available. I assume the use of such an adaptive algorithm and use the term "phase space generator" to refer to the mapping from the hypercube to phase space, along with a formula for the weight factor.

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Most of the traditional literature on the subject [2, 3] concerns itself with the general problem of generating phase space distributions for particles with arbitrary masses. In the context of present-day (and planned) colliders, however, most of the "final-state" particles (quarks and leptons) are massless or nearly so, compared to the typical momentum transfers in processes of interest. This was emphasized by Kleiss, Stirling, and Ellis [4], who presented a phase generator, Rambo, intended for this regime.

The Rambo generator first generates an isotropic set of massless four-momenta *not* satisfying energy-momentum conservation. Afterwards, it applies a Lorentz transformation to obtain a momentum-conserving, and then a conformal transformation to obtain an energy and momentum-conserving, set of massless four-momenta. (For phase space with massive particles, another scaling of the momenta and recalculation of the energies yields a valid configuration.) In this way, the $3n-4$ independent variables describing a final state with n massless particles are smeared smoothly over $3n$ variables (which are in turn mapped into a uniform distribution over $4n$ variables). This generator is both elegant and simple to program, which makes it an extremely useful check on more complicated generators such as the one I present below.

Unlike traditional generators, the Rambo generator has a weight factor which (for the purely massless case) is independent of the point in phase space; it contributes a factor dependent only on the total center-of-mass energy in the process. Thus in a certain formal sense it has "maximal efficiency": a uniform weight over the integration region will yield the minimum error (for a given amount of computational work) in a Monte Carlo integration procedure. Indeed, if we were simply interested in calculating the volume of phase space, this generator would be unsurpassable in efficiency. Practical applications differ in two respects: we wish to integrate a scattering matrix element; and we wish to perform the integration over that part of phase space that survives certain angle and energy cuts. For such practical applications, the efficiency of the Rambo generator is not maximal, and one can improve upon it.

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2. MONTE CARLO INTEGRATION

We are interested in calculating a cross section for the production of n final-state particles,

$$\sigma_s(n)=\int_{\substack{\text{phase space}\\ \text{with cuts}}} d\mathrm{LIPS}_n(P;\{p_i,m_i\})\,FS \times |\mathscr{A}_{n+2}(P;\{p_i,m_i\})|^2, \tag{2.1}$$

where $\mathscr{A}_{n+2}$ is the scattering amplitude, F is the flux factor for the incoming particles, S the symmetry factors for the final state, P is the sum of the four-momenta of the incoming particles (with the convention that $E_{\text{incoming}}>0$), and where LIPS is the Lorentz-invariant phase space measure for n particles with four-momenta p_i and masses m_i:

$$\begin{aligned} d\mathrm{LIPS}(P;\{p_i,m_i\}) &= (2\pi)^4\,\delta^4\left(P-\sum_{i\in\text{final}} p_i\right) \times \prod_{i\in\text{final}} \frac{d^4p_i}{(2\pi)^4}\,2\pi\delta(p_i^2-m_i^2)\,\theta(p_i^0) \\ &= (2\pi)^4\,\delta^4\left(P-\sum_{i\in\text{final}} p_i\right) \times \prod_{i\in\text{final}} \frac{d^3\mathbf{p}_i}{(2\pi)^3\,2E_i}. \end{aligned} \tag{2.2}$$

I have suppressed any additional integrations that may arise (such as the integration over parton distributions in the case of hadron–hadron collisions). We are particularly interested in the light (or massless) particle case, where $E_i \simeq |\mathbf{p}_i|$ for momenta surviving the cuts. A phase space generator provides us with a mapping from the hypercube to Lorentz-invariant phase space, $p_i=G_i(x)$, and the Jacobian of the transformation, $W(x)$. The cross section of interest is then

$$\begin{aligned} \sigma_s(n) &= \int_{[0,1]^{d(n)}} dx\,\theta_{\text{cut}}(\{G_i(x)\})\,W(x)\,FS \times |\mathscr{A}_{n+2}(P;\{G_i(x),m_i\})|^2 \\ &\equiv \int_{[0,1]^{d(n)}} dx\,\mathscr{M}(x), \end{aligned} \tag{2.3}$$

where

$$\theta_{\text{cut}}(\{p_i\})=\begin{cases}1 & \text{if the set } p_i \text{ passes the cuts;}\\ 0 & \text{otherwise.}\end{cases} \tag{2.4}$$

If we choose N_{MC} points in our Monte Carlo sample, with probability density $\mathscr{P}(x)$, the estimate of the cross section is given by [1]

$$\sigma_{\text{MC}}=\frac{1}{N_{\text{MC}}}\sum_x \frac{\mathscr{M}(x)}{\mathscr{P}(x)} \tag{2.5}$$

and the fractional error estimate is given (for large N_{MC}) by

$$\varepsilon=\frac{1}{\sigma_{\text{MC}}}\sqrt{(v_{\text{MC}}-(\sigma_{\text{MC}})^2)/(N_{\text{MC}}-1)}, \tag{2.6}$$

where

$$v_{\text{MC}}=\frac{1}{N_{\text{MC}}}\sum_x\left(\frac{\mathscr{M}(x)}{\mathscr{P}(x)}\right)^2. \tag{2.7}$$

In the simplest implementation, the probability distribution would be uniform ($\mathscr{P}(x)=1$); more sophisticated approaches, such as the Vegas algorithm, attempt to choose the probability distribution so as to minimize the error.

In performing the calculations, we wish to minimize the amount of work required to obtain a specified accuracy in our calculations; or equivalently, to maximize the accuracy obtained for a given amount of work. In numerical calculations, this translates into the ideal of minimizing the amount of computer time required to obtain an answer to a specified degree of accuracy. In Monte Carlo calculations, the computer time is proportional to the number of points taken; and the error (in the limit of a large number of points) decreases in proportion to the square root of the number of points. The most hard-nosed measure of efficiency is thus a quantity like 1/(computer time × (relative error)^2), which I shall term the *practical efficiency*. In the limit of a large number of Monte Carlo points, the practical efficiency approaches a constant for any given calculation.

However, this constant depends on the details of the hardware and software system. It is therefore perhaps preferrable to think about the ordinary efficiency, which I define

$$e=\frac{1}{N_{\text{MC}}\varepsilon^2}, \tag{2.8}$$

where N_{MC} is the number of Monte Carlo points used, and ε is the fractional error in the answer (as estimated, for example by Vegas). In addition, it will be helpful to define a *hit rate* h, the fraction of points thrown down by Vegas that survive the quasi-experimental cuts.

A hit rate close to unity is desirable; in that case, the phase space generator spends most of its time generating useful points, rather than points to be discarded. One might assume that the amount of time spent generating phase space configurations is in any event negligible compared to the amount of time spent evaluating the matrix element for a scattering process. This assumption is incorrect for two reasons: first, for some background processes, such as multijet production, there are numerically reasonable approximations [5], which are also reasonably efficient computationally; second, the volume of phase space which survives the cuts typically decreases *factorially* with an increasing number of final-state particles,[1] whereas the approximations to scattering matrix elements can often be cast in forms where the amount of computational work increases only polynomially [6]. In such a case, if the hit rate is proportional to the fractional volume of phase space that survives the cuts, then the computation time for a large number of final-state particles is dominated by the phase space generator, even if it is much faster to generate a single configuration than to evaluate a scattering matrix element.

Vegas will attempt to increase the hit rate by performing changes of variables numerically, and thereby adjusting the distribution $\mathscr{P}(x)$, but it will be able to do so only in cases where the cuts are approximately parallel to the axes of the hypercube. Rambo, however, effectively smears the Lorentz-invariant phase space over the hypercube in such a way that the cuts depend non-trivially on all the variables. Vegas is then unable to improve the hit rate much by re-mapping the coordinates of its hypercube; indeed, when driving Rambo from Vegas, one typically sees a hit rate comparable to the fractional volume of phase space that survives the cuts.

The above considerations are in some ways a special case[2] of the general observation [1] that in an importance-sampled Monte Carlo integration, one obtains an optimal choice for the distribution of points by taking

$$\mathscr{P}(x)=\frac{|\mathscr{M}(x)|}{\int dx\,|\mathscr{M}(x)|}. \tag{2.9}$$

Vegas attempts to make this choice by changing variables numerically. If we can find a phase space mapping that allows Vegas to choose the probability distribution more effectively, we will improve our efficiency.

Even if we can do this, however, adaptive algorithms often have trouble handling singular or sharply-peaked behavior in the integrand. If possible, it is preferrable to absorb singularities (or even cut-off singularities) into the probability distribution *analytically*. In the case of interest, scattering amplitudes for massless particles typically exhibit two sorts of singularities. The matrix element will diverge as the any outgoing particle becomes soft; and as any two outgoing particles become collinear. As we shall see, one can absorb the former singularity into the probability distribution, thereby smoothing out the integrand and improving the efficiency of our integration.

A generator which increases the hit rate over that obtainable with Rambo, and absorbs some of the singularities of the matrix element, is given in the next few sections. I call it Octopus.

[1] This is typical of the dimension depndence of the ratio of volumes of a regular solid embedded in another of different shape.

[2] The considerations are not precisely the same, because the computer time for evaluating configurations which do or do not survive the cuts is different.

3. MASSLESS PHASE SPACE WITH CUTS

Let us begin by focussing attention on the case of a phase space generator for massless particles; we shall consider the more general case of light particles in Section 4. What are the sorts of simulated experimental cuts one may wish to apply? In electron–positron colliders, the lab frame and center-of-mass frame are the same, and so the appropriate cuts are minimum energy cuts $E_{\min}$ (to eliminate soft junk), minimum angle cuts $\theta_{\min}$ between outgoing particles, and a minimum angle cut θ_{beam} with respect to the beam direction (to exclude debris travelling down the beam pipe). In hadron–hadron colliders, collisions involve partons of varying energies, and so the center-of-mass frames of different collisions are smeared along the beam direction; in this case, a transverse energy cut $E_{\text{T}\min}$ is appropriate. (The transverse energy is defined as the projection of the energy onto the plane transverse to the beam; for a massless particle, it is the same as the transverse momentum.) In addition, experimenters impose cuts on the pseudo-rapidities of jets and on the cone angle ΔR between jets. We shall mimic these in the conventional manner with limits on the maximum pseudo-rapidities, $\eta=-\ln(\tan\theta/2)$ (which for massless particles is the same as the rapidity $y=\ln[(E+\mathbf{p}_{\|})/(E-\mathbf{p}_{\|})]/2$), of the outgoing partons, and with limits on the minimum ΔR,

$$\Delta R=\sqrt{(\Delta\eta)^2+(\Delta\phi)^2} \tag{3.1}$$

between outgoing partons (I assume below that $\Delta R_{\min}\leqslant\pi/2$). Of course, these sorts of cuts are also necessary in a theoretical calculation in order to cut off the infrared divergences of scattering amplitudes.

To simplify the presentation, I consider only cuts that treat all particles symmetrically; this is in fact true of the cuts imposed in one physically important situation, the calculation of multijet cross sections. It is, however, possible to generalize the equations presented below to different cuts for different outgoing particles, so long as the cuts are of the same general types given above. When using the Rambo algorithm, the cuts are applied afterward to each momentum set generated. The event is rejected if the set fails the cuts. I assumed a similar check is applied to the output of the generator described below, so that we may (if desired) apply a weaker set of constraints within the generator. That will not cause us to generate sets of momenta that fail the desired cuts, but will only reduce the efficiency. Thus we need not solve the constraints implied by the cuts exactly, but only approximately.

One can write an iterative formula for the phase space measure [2],

$$dLIPS_n(P_{tot}; \{p_i\}_{i=1}^n) = \frac{d^3\mathbf{p}_1}{(2\pi)^3\, 2E_1}\, dLIPS_{n-1}(P_{tot}-p_1; \{p_i\}_{i=2}^n) \tag{3.2}$$

which I shall use as the basis for the algorithm.

For massless particles, Eq. (3.2) becomes

$$\begin{aligned} dLIPS_n(P_{tot}; \{p_i\}_{i=1}^n) &= \frac{1}{2(2\pi)^3} E_1\, dE_1\, d\phi_1\, d\cos\theta_1 \\ &\times dLIPS_{n-1}(P_{tot}-p_1; \{p_i\}_{i=2}^n), \end{aligned} \tag{3.3}$$

where it will be convenient to take θ_i and ϕ_i as the polar and azimuthal angle, respectively, of the ith particle with respect to the beam axis. I utilize a hypercube of dimension $d(n) = 3n-4$ (in contrast to the Rambo's $d(n) = 4n$), with the following correspondences for the first $n-2$ momenta,

$$\begin{aligned} x_{3i-2} &\leftrightarrow E_i \\ x_{3i-1} &\leftrightarrow \theta_i \\ x_{3i} &\leftrightarrow \phi_i \end{aligned} \tag{3.4}$$

with each $x \in (0, 1)$.

What are the integration limits on these variables? Let us assume that we have generated momenta for particles $1, ..., (i-1)$ and define the remaining four-momentum, $P = P_{tot} - \sum_{j=1}^{i-1} p_i$. We must ensure that, after generating a momentum for the first remaining particle, we will be able to satisfy the energy-momentum conservation constraint for the other remaining particles. Now, the sum of any number of four-momenta with positive energies is a positive-mass four-momentum, so we must require that

$$(P-p_i)^2 \geqslant 0. \tag{3.5}$$

This constraint is also clearly sufficient to satisfy energy-momentum conservation, since one can always write a positive mass squared four-momentum in terms of two massless four-momenta (setting the rest to 0). Thus

$$2E_i(P^0 - |\mathbf{P}| \cos\theta_{iP}) \leqslant (P^0)^2 - |\mathbf{P}|^2, \tag{3.6}$$

where θ_{iP} is the angle between $\mathbf{p}_i$ and $\mathbf{P}$, so that defining the dimensionless quantities

$$\begin{aligned} e_i &= E_i/P^0 \\ v &= |\mathbf{P}|/P^0 \\ e_{T\min} &= E_{T\min}/P^0 \\ e_{\min} &= E_{\min}/P^0, \end{aligned} \tag{3.7}$$

we have the constraints

$$\begin{aligned} e_i &\leqslant \frac{1+v}{2} \\ \cos\theta_{iP} &\geqslant \frac{2e_i + v^2 - 1}{2e_i v} \equiv L_{iP}. \end{aligned} \tag{3.8}$$

(The kinematic limit on e_i ensures that $L_{iP} \leqslant 1$.) As detailed in Appendix I, the latter constraint implies the following constraints on $\cos\theta_i$ and ϕ_i,

$$\begin{aligned} \cos\theta_i &\in [c^-, c^+] \\ \phi_i &\in [\phi^-, \phi^+], \end{aligned} \tag{3.9}$$

where

$$\left.\begin{aligned} c^- &= -1 \\ c^+ &= 1 \\ \phi^- &= \phi_{\mathbf{P}} - \pi \\ \phi^+ &= \phi_{\mathbf{P}} + \pi \end{aligned}\right\}, \quad L_{iP} \leqslant -1$$

$$\left.\begin{aligned} c^- &= \begin{cases} -1, & L_{iP} \leqslant -\cos\theta_{\mathbf{P}} \\ L_{iP}^-, & \text{otherwise} \end{cases} \\ c^+ &= \begin{cases} 1, & L_{iP} \leqslant \cos\theta_{\mathbf{P}} \\ L_{iP}^+, & \text{otherwise} \end{cases} \\ \phi^- &= \phi_{\mathbf{P}} - L_\phi \\ \phi^+ &= \phi_{\mathbf{P}} + L_\phi \end{aligned}\right\}, \quad L_{iP} > -1 \tag{3.10}$$

and where $\theta_{\mathbf{P}}$ is the polar angle of $\mathbf{P}$, and

$$\begin{aligned} L_{iP}^{\pm} &= L_{iP} \cos\theta_{\mathbf{P}} \pm \sqrt{(1-L_{iP}^2)(1-\cos^2\theta_{\mathbf{P}})} \\ L_\phi &= \operatorname{acos}\left(\frac{L_{iP} - \cos\theta_i \cos\theta_{\mathbf{P}}}{\sqrt{(1-\cos^2\theta_i)(1-\cos^2\theta_{\mathbf{P}})}}\right). \end{aligned} \tag{3.11}$$

For the electron–positron case we also have the constraints $e_i \geqslant e_{\min}$ and $|\cos\theta_i| \leqslant \cos\theta_{\text{beam}}$. We must in addition leave sufficient energy for the minimums of the remaining particles, and thus

$$e_i \leqslant \min\left(\frac{1+v}{2}, 1-(n-i)\, e_{\min}\right). \tag{3.12}$$

In the hadron–hadron case, we can translate the constraint $|\eta_i| \leqslant \eta_{\max}$ into a constraint

$$|\cos\theta_i| \leqslant \cos\theta_{\text{beam}} \tag{3.13}$$

by defining $\cos\theta_{\text{beam}} = \tanh\eta_{\max}$. Every particle must have an energy greater than or equal to the minimum transverse energy, so we have the upper bound

$$e_i \leqslant \min\left(\frac{1+v}{2}, 1-(n-i)\, e_{T\min}\right) \equiv e_{\max}. \tag{3.14}$$

The transverse energy constraint $e_i \sin\theta_i \geqslant e_{\rm T\,min}$ itself imposes the additional constraint on $\cos\theta_i$,

$$|\cos\theta_i| \leqslant \sqrt{1-(e_{\rm T\,min}/e_i)^2} \tag{3.15}$$

(given that $e_i \geqslant e_{\rm T\,min}$, of course). To ensure that this has a non-trivial overlap with the interval $[c^-, c^+]$ given in Eq. (3.10), we must demand that

$$\begin{aligned} \sqrt{1-(e_{\rm T\,min}/e_i)^2} &> c^- \\ -\sqrt{1-(e_{\rm T\,min}/e_i)^2} &< c^+. \end{aligned} \tag{3.16}$$

Although it is in principle possible to impose these constraints exactly, the non-trivial cases turn out to involve the solutions to a quartic equation. As discussed in Appendix II, it is therefore preferrable to impose a slightly weaker set of constraints,

$$\begin{aligned} &e_i \geqslant e_{\rm T\,min} \\ &e_i \geqslant L_{ET}, \qquad \text{if} \quad e_{\rm T\,min} > \frac{1-v^2}{2(1-v\sin\theta_{\rm P})} \\ &\qquad\qquad \text{and } 4\alpha e_{\rm T\,min}^2 + \beta^2(1-v^2)\cos^2\theta_{\rm P} \geqslant 0, \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} \alpha &= \sin^2\theta_{\rm P} - (1-v^2)\cos^2\theta_{\rm P} \\ \beta &= \sqrt{1-v^2}\cos\theta_{\rm P} - v\sin\theta_{\rm P} \\ L_{ET} &= \frac{1}{2} + \frac{v(\beta\sin\theta_{\rm P} + \sqrt{4\alpha e_{\rm T\,min}^2 + \beta^2(1-v^2)\cos^2\theta_{\rm P}})}{2\alpha}. \end{aligned} \tag{3.18}$$

(The prerequisites on $e_{\rm T\,min}$ are nearly always true in practice for those situations where the second constraint is more severe than the first.)

We also want to ensure that the choice of energy for p_i still allows us to satisfy the transverse energy constraint for the following $n-i$ momenta. That constraint, combined with the requirement of energy-momentum conservation, implies that there is a maximum *longitudinal* momentum that the following momenta can have; and thus, we must not allow the present p_i to increase the existing longitudinal momentum too much. That is, we must demand

$$\max \sum_{j=i+1}^{n} |p_{jL}| \geqslant |P_L - p_{iL}|. \tag{3.19}$$

With

$$\begin{aligned} \omega &= 1 - v^2\cos^2\theta_{\rm P} - (n-i)^2 e_{\rm T\,min}^2 + e_{\rm T\,min}^2 \\ \chi &= 1 - \frac{e_{\rm T\,min}^2(1-v^2\cos^2\theta_{\rm P})}{\omega^2} \\ L_{PL}^{\pm} &= \frac{\omega}{2(1-v^2\cos^2\theta_{\rm P})}(1 \pm v\,|\cos\theta_{\rm P}|\,\sqrt{\chi}), \end{aligned} \tag{3.20}$$

this constraint, as shown in Appendix III, translates into the requirement that

$$\begin{aligned} e_i &\leqslant L_{PL}^+, \qquad \chi > 0 \\ e_i &\leqslant \omega/2, \qquad \chi < 0, \end{aligned} \tag{3.21}$$

so long as

$$\begin{aligned} &e_{\rm T\,min}^2 + v^2\cos^2\theta_{\rm P} > e_{\rm max}^2 \\ &L_{PL}^- \leqslant \max(\omega/2, e_{\rm T\,min}, L_{ET}) \qquad \text{or} \qquad \chi < 0. \end{aligned} \tag{3.22}$$

These prerequisites on $e_{\rm T\,min}$ are again nearly always true in practice. (One would omit the constraints of Eq. (3.21) if they were not.)

Given e_i, we also obtain an addition constraint on $\cos\theta_i$,

$$\cos\theta_i \in \left[\frac{v\cos\theta_{\rm P} - \sqrt{(1-e_i)^2 - (n-i)^2 e_{\rm T\,min}^2}}{e_i}, \frac{v\cos\theta_{\rm P} + \sqrt{(1-e_i)^2 - (n-i)^2 e_{\rm T\,min}^2}}{e_i}\right]. \tag{3.23}$$

Equations (3.14), (3.17), and (3.21) together give upper and lower limits $e_{\rm l}$ and $e_{\rm u}$ on the energy fraction of the particle. To generate an energy from the corresponding x, we could set

$$E_i = P^0(x_{3i-2}(e_{\rm u} - e_{\rm l}) + e_{\rm l}) \tag{3.24}$$

with an associated jacobian $e_{\rm u} - e_{\rm l}$. However, as noted in the previous section, massless-particle amplitudes have soft singularities of the form $\mathcal{M}(x) \sim 1/E^2$, while the measure in Eq. (3.3) only has one power of the energy. In order to absorb the remaining singularity (and thereby introduce another power of the energy into the measure), we instead should set

$$E_i = P^0 e_{\rm l} \exp[\ln(e_{\rm u}/e_{\rm l})\, x_{3i-2}]; \tag{3.25}$$

the associated jacobian is then

$$J_i^E = E_i \ln(e_{\rm u}/e_{\rm l}). \tag{3.26}$$

For some purposes (for example, computing the correlation between different amplitudes), one may desire an even larger explicit power of E in the measure. To obtain a net power of E_i^{q+2}, we should set

$$E_i = P^0 e_{\rm l} \frac{e_{\rm u}}{(e_{\rm u}^q - (e_{\rm u}^q - e_{\rm l}^q)\, x_{3i-2})^{1/q}}. \tag{3.27}$$

The associated jacobian in that case would be

$$J_i^E = E_i^{q+1}\,\frac{e_u^q - e_l^q}{q e_u^q e_l^q (P^0)^q}. \tag{3.28}$$

Given the value of e_i, Eqs. (3.10), (3.13), (3.15), and (3.23) together give upper and lower bounds c_u and c_l on $\cos\theta_i$. We can then set

$$\cos\theta_i = (c_u - c_l)\,x_{3i-1} + c_l. \tag{3.29}$$

The associated jacobian is

$$J_i^\theta = (c_u - c_l)/2. \tag{3.30}$$

Before turning to the question of satisfying the interparton angle constraints themselves, we may note that the very existence of such constraints forces the minimum invariant mass of a pair of final-state particles to be greater than some minimum,

$$(p_i + p_j)^2 \geqslant 2E_{T\min}^2(1 - \cos\Delta R_{\min}) \equiv m_{\text{pair}}^2. \tag{3.31}$$

(This formula holds for the case of hadron–hadron scattering; there is, of course, a similar one for the electron–positron case.) This allows us to replace Eq. (3.5) with a stronger constraint,

$$\begin{aligned}(P - p_i)^2 &\geqslant N_{\text{pairs}} m_{\text{pair}}^2 \\ &= \frac{(n-i)(n-i-1)}{2}\, m_{\text{pair}}^2.\end{aligned} \tag{3.32}$$

With $\mu_{\text{pairs}}^2 = N_{\text{pairs}} m_{\text{pair}}^2/(P^0)^2$, the maximum energy fraction is then reduced a bit,

$$e_i \leqslant \frac{1+v}{2} - \frac{\mu_{\text{pairs}}^2}{2(1-v)}, \tag{3.33}$$

and the relative cosine limit becomes a bit tighter for the given energy fraction,

$$\cos\theta_{iP} \geqslant \frac{2e_i + v^2 - 1 + \mu_{\text{pairs}}^2}{2e_i v} \equiv L'_{iP}. \tag{3.34}$$

Equation (3.10) continues to hold, with $L_{iP} \to L'_{iP}$. However, since this modification makes the cosine constraint stronger, we can continue to use the (weaker) constraints following from Eq. (3.16); they will not (improperly) eliminate any configurations which would survive the cuts.

Let us now examine the question of satisfying the cone angle constraints, $\Delta R_{ij} \geqslant \Delta R_{\min}$. (Although I shall not discuss it in detail, the method of satisfying the angular separation constraints in an electron–positron environment is similar in many ways.) It is convenient to do this by ignoring any potential constraints on θ_i and imposing constraints only on ϕ_i. This will result in a less-than-maximal hit rate, but in a hadron–hadron environment this choice is acceptable, since with the usual definition of ΔR the cones are wider in the azimuthal angle than in the polar one. We wish to exclude all angles ϕ_i for which $\Delta R_{ij} < \Delta R_{\min}$ for any $j < i$. The remaining angles, in general, form a disjoint set of intervals. How should we attack this problem?

The method I describe is, of course, not the only possible one, but it is convenient. The idea is to subdivide the circle $[0, 2\pi]$ into some number of wedges and to iterate a marking process over all $j < i$. For any given j, one marks as excluded all wedges which lie entirely within $\Delta R_{\min}$ of the jth particle. One then generates an angle uniformly within the unmarked wedges; the associated weight factor is simply the number of unmarked wedges divided by the total number of wedges.

In practice, we might as well subdivide the interval $[\phi^-, \phi^+]$ rather than the whole circle into, say, B bins, numbered 0, ..., $B-1$. (It is most convenient to choose B to be a multiple of the number of bits in a computer word and to let each flag for a wedge be represented by a single bit. For practical purposes, $B = 96$ and $B = 128$ are good choices.) Shifting all angles by $\phi_{\mathbf{P}}$ simplifies matters somewhat; define

$$\begin{aligned} r_{i,j} &= \Delta R_{\min}^2 - (\eta_i - \eta_j)^2 \\ \varphi_j^\pm &= (\phi_j - \phi_{\mathbf{P}} \pm \sqrt{r_{i,j}}) \bmod 2\pi \\ b_l &= \left\lceil \frac{(\varphi_j^- + L_\phi)B}{2L_\phi} \right\rceil \\ b_u &= \left\lfloor \frac{(\varphi_j^+ + L_\phi)B}{2L_\phi} - 1 \right\rfloor, \end{aligned} \tag{3.35}$$

where "mod 2π" means shifting into the interval $(-\pi, \pi]$ by adding or subtracting an appropriate multiple of 2π. (The increment of -1 in the b_u is intended to ensure that the entire bin falls within the excluded region.) Note that the assumed limit on $\Delta R_{\min}$ implies that $r_{i,j} \leqslant \pi^2/4$.

All bins are initiall marked as allowed. We must iterate the following steps for each $j < i$ for which $r_{i,j} > 0$:

$$\begin{aligned} &\text{if } \varphi_j^- \leqslant \varphi_j^+, && \text{mark bins } b_l \cdots b_u \text{ as excluded} \\ &\text{if } \varphi_j^- > \varphi_j^+, && \text{mark bins } b_l \cdots B-1 \\ & && \text{and bins } 0 \cdots b_u \text{ as excluded.} \end{aligned} \tag{3.36}$$

(It should be understood that "marking" a bin with a number less than zero or greater than $B-1$ has no effect, and that the sequence $a_1 \cdots a_2$ is empty if $a_1 > a_2$. The inequality on the $\varphi_j^{\pm}$, rather than the more obvious one on the $b_{l,u}$, ensures that we do not exclude a bin if the entire excluded region falls within the bin.) This will leave us with a set of B_a allowed bins, which we label $b_1, ..., b_{B_a}$. (If none of the bins are allowed, reject the configuration.) The angle ϕ_i is then given by

$$\begin{aligned} k_\phi &= \lfloor B_a x_{3i} \rfloor \\ b_\phi &= b_{k_\phi} \\ \delta_\phi &= B_a x_{3i} - k_\phi \\ \phi_i &= \phi_{\mathbf{P}} - L_\phi + 2L_\phi \frac{b_\phi + \delta_\phi}{B}. \end{aligned} \tag{3.37}$$

The jacobian associated with the generation of ϕ_i is

$$J_i^\phi = \frac{L_\phi}{\pi} \frac{B_a}{B}. \tag{3.38}$$

The astute reader will note that this part of the algorithm has a running time which scales quadratically with the number of outgoing particles, rather than linearly as do the remaining pieces. This may seem bad in contrast to Rambo, whose running time scales linearly with the number of particles, but in fact this difference is somewhat of an illusion, because the running time to check whether an event passes the cuts also scales quadratically with the number of particles.

Thus far the discussion has concerned the first $n-2$ final-state four-momenta. For the last two, we must do things a bit differently, because we have only two independent variables in total, rather than three per particle. The phase space measure for these two particles is

$$\frac{dE_{n-1}}{4v} d\phi_{n-1} = e_{n-1}^2 \frac{d\cos\theta_{n-1}}{2(1-v^2)} d\phi_{n-1}. \tag{3.39}$$

I shall choose as the independent variables the cosine of the polar angle of the $(n-1)$th particle, and the azimuthal angle of this particle, both with respect to the sum of the $(n-1)$th and nth momenta. This azimuthal angle is then unaffected by energy-momentum conservation constraints and is constrained only by the additional sorts of constraints considered above.

Thus I define a new coordinate system, with

$$\begin{aligned} \hat{\mathbf{e}}_1 &= \hat{\mathbf{P}} \\ \hat{\mathbf{e}}_2 &= \frac{\hat{\mathbf{z}} \times \hat{\mathbf{P}}}{|\hat{\mathbf{z}} \times \hat{\mathbf{P}}|} = -\sin\phi_{\mathbf{P}} \hat{\mathbf{x}} + \cos\phi_{\mathbf{P}} \hat{\mathbf{y}} \\ \hat{\mathbf{e}}_3 &= \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = (\hat{\mathbf{z}} - \cos\theta_{\mathbf{P}} \hat{\mathbf{P}})/\sin\theta_{\mathbf{P}}. \end{aligned} \tag{3.40}$$

(At this point, $P = P_{\text{tot}} - \sum_{j=1}^{n-2} p_i$.) For the two last particles, we take θ_{n-1} and θ_n to be the polar angles with respect to the $\hat{\mathbf{e}}_1$ axis, and ϕ_{n-1}, ϕ_n to be the azimuthal angles in the $\hat{\mathbf{e}}_2$–$\hat{\mathbf{e}}_3$ plane (with $\phi = 0$ in the $\hat{\mathbf{e}}_2$ direction). It will be convenient to define a unit vector in the direction (or opposite to the direction) of the projection of $\hat{\mathbf{e}}_3$ into the original x–y plane,

$$\hat{\mathbf{e}}_4 = \sin\theta_{\mathbf{P}} \hat{\mathbf{e}}_1 - \cos\theta_{\mathbf{P}} \hat{\mathbf{e}}_3. \tag{3.41}$$

The minimum energy constraints immediately imply the range for e_{n-1},

$$e_{n-1} \in \left[e_l' \equiv \max\left(e_{\text{T}\min}, \frac{1-v}{2}\right), \; e_u' \equiv \min\left(1 - e_{\text{T}\min}, \frac{1+v}{2}\right) \right], \tag{3.42}$$

for hadron–hadron scattering, with analogous limits in the electron–positron case.

We can solve for the cosines of the angles in terms of the energy fractions of the particles:

$$\begin{aligned} \cos\theta_{n-1} &= \frac{e_{n-1}^2 - e_n^2 + v^2}{2e_{n-1} v} \\ &= \frac{v^2 + 2e_{n-1} - 1}{2e_{n-1} v} \\ \cos\theta_n &= \frac{e_n^2 - e_{n-1}^2 + v^2}{2e_n v} \\ &= \frac{v^2 - 2e_{n-1} + 1}{2(1 - e_{n-1}) v} \end{aligned} \tag{3.43}$$

and thereby arrive at limits for $\cos\theta_{n-1}$,

$$\begin{aligned} c_l' &= \max\left(-1, \frac{v^2 + 2e_l' - 1}{2e_l' v}\right) \\ c_u' &= \min\left(1, \frac{v^2 + 2e_u' - 1}{2e_u' v}\right). \end{aligned} \tag{3.44}$$

To generate $\cos\theta_{n-1}$, we set

$$\cos\theta_{n-1} = (c_u' - c_l')\, x_{3n-5} + c_l' \tag{3.45}$$

with the corresponding jacobian,

$$J_{n-1}^\theta = (c_u' - c_l') \frac{2e_{n-1}^2}{1 - v^2}, \tag{3.46}$$

where we have put the factors from the measure into the jacobian for convenience. (See the comments on numerical stability at the end of the following section for an alternate version of this equation in certain corners of phase space.) The two energy fractions and the other cosine are then

$$\begin{aligned} e_{n-1} &= \frac{1-v^2}{2(1-v\cos\theta_{n-1})} \\ e_n &= \frac{1-2v\cos\theta_{n-1}+v^2}{2(1-v\cos\theta_{n-1})} = 1-e_{n-1} \\ \cos\theta_n &= \frac{2v-(1+v^2)\cos\theta_{n-1}}{1-2v\cos\theta_{n-1}+v^2}. \end{aligned} \tag{3.47}$$

The transverse energy fractions of the particles are

$$\begin{aligned} e^2_{\mathrm{T}n-1} &= (\mathbf{p}_{n-1}\cdot\hat{\mathbf{e}}_2)^2 + (\mathbf{p}_{n-1}\cdot\hat{\mathbf{e}}_4)^2 \\ &= e^2_{n-1}[\sin^2\theta_{n-1}\cos^2\phi_{n-1} \\ &\quad + (\cos\theta_{n-1}\sin\theta_{\mathbf{P}} \\ &\quad - \sin\theta_{n-1}\sin\phi_{n-1}\cos\theta_{\mathbf{P}})^2] \\ &= e^2_{n-1}[\sin^2\theta_{n-1} + (\cos^2\theta_{n-1} \\ &\quad - \sin^2\theta_{n-1}\sin^2\phi_{n-1})\sin^2\theta_{\mathbf{P}} \\ &\quad - 2\cos\theta_{n-1}\sin\theta_{n-1}\sin\theta_{\mathbf{P}}\cos\theta_{\mathbf{P}}\sin\phi_{n-1}]. \end{aligned} \tag{3.48}$$

The transverse energy constraint then is

$$\begin{aligned} &\sin^2\theta_{n-1}\sin^2\theta_{\mathbf{P}}\sin^2\phi_{n-1} \\ &\quad + 2\cos\theta_{n-1}\sin\theta_{n-1}\sin\theta_{\mathbf{P}}\cos\theta_{\mathbf{P}}\sin\phi_{n-1} \\ &\quad - \sin^2\theta_{n-1} - \cos^2\theta_{n-1}\sin^2\theta_{\mathbf{P}} \\ &\quad + \left(\frac{e^2_{\mathrm{T\,min}}}{e_{n-1}}\right)^2 < 0 \end{aligned} \tag{3.49}$$

or

$$\sin\phi_{n-1} \in [s^-_{n-1}, s^+_{n-1}], \tag{3.50}$$

where

$$s^\pm_i = \frac{-\cos\theta_{\mathbf{P}}\cos\theta_i \pm \sqrt{1-e^2_{\mathrm{T\,min}}/e^2_i}}{\sin\theta_{\mathbf{P}}\sin\theta_i}. \tag{3.51}$$

Now, $\sin\phi_n = -\sin\phi_{n-1}$, so we obtain another restriction,

$$\sin\phi_{n-1} \in [-s^+_n, -s^-_n]. \tag{3.52}$$

Combining the two and defining

$$\begin{aligned} s_1 &= \max(-1, -s^+_n, s^-_{n-1}) \\ s_{\mathrm{u}} &= \min(1, s^+_{n-1}, -s^-_n), \end{aligned} \tag{3.53}$$

we obtain

$$\sin\phi_{n-1} \in [\max(-1, -s^+_n, s^-_{n-1}), \min(1, s^+_{n-1}, -s^-_n)]. \tag{3.54}$$

It will be helpful to define

$$\begin{aligned} A_1 &= \operatorname{asin} s_1 \\ A_{\mathrm{u}} &= \operatorname{asin} s_{\mathrm{u}}. \end{aligned} \tag{3.55}$$

For the ΔR constraints involving the final two particles, one could in principle proceed along the same lines as above; but in this case, the rapidity and azimuthal angle (in the lab frame) of the particles are non-polynomial functions of $\sin\phi_{n-1}$, and thus implementing that constraint would require solving many equations numerically, which is likely to be rather expensive. Instead, it is easier to implement the somewhat weaker constraint excluding not the full circle, but only the circumscribed square $|\Delta\eta| \leqslant \Delta R_{\min}/\sqrt{2}$, $|\Delta\phi| \leqslant \Delta R_{\min}/\sqrt{2}$. The remaining cuts will then be applied after phase space generation, as usual. Denote by θ^L_{n-1} and ϕ^L_{n-1} the polar and azimuthal angles of the $(n-1)$th particle in the lab coordinate system $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$. Then

$$\begin{aligned} \cos\theta^L_{n-1} &= \cos\theta_{n-1}\cos\theta_{\mathbf{P}} \\ &\quad + \sin\theta_{n-1}\sin\theta_{\mathbf{P}}\sin\phi_{n-1} \\ \sin\theta^L_{n-1}\sin\phi^L_{n-1} &= \cos\theta_{n-1}\sin\theta_{\mathbf{P}}\sin\phi_{\mathbf{P}} \\ &\quad + \sin\theta_{n-1}\cos\phi_{n-1}\cos\phi_{\mathbf{P}} \\ &\quad - \sin\theta_{n-1}\sin\phi_{n-1}\cos\theta_{\mathbf{P}}\sin\phi_{\mathbf{P}} \\ \sin\theta^L_{n-1}\cos\phi^L_{n-1} &= \cos\theta_{n-1}\sin\theta_{\mathbf{P}}\cos\phi_{\mathbf{P}} \\ &\quad - \sin\theta_{n-1}\cos\phi_{n-1}\sin\phi_{\mathbf{P}} \\ &\quad - \sin\theta_{n-1}\sin\phi_{n-1}\cos\theta_{\mathbf{P}}\cos\phi_{\mathbf{P}}. \end{aligned} \tag{3.56}$$

Equation (3.54) specifies two regions in ϕ_{n-1}, where $\sin\phi_{n-1}$ satisfies the given constraint and where $\cos\phi_{n-1}$ is either positive (R_+) or negative (R_-). Divide these regions into bins. For each $i < n-1$, we may form an allowed set in ϕ_{n-1}, consisting of those bins which satisfy one of the equations

$$\begin{aligned} \operatorname{atanh}\cos\theta^L_{n-1} &> \eta_i + \Delta R_{\min}/\sqrt{2} \\ \operatorname{atanh}\cos\theta^L_{n-1} &< \eta_i - \Delta R_{\min}/\sqrt{2} \\ \phi^L_{n-1} - \phi_i &> \Delta R_{\min}/\sqrt{2} \\ \phi^L_{n-1} - \phi_i &< -\Delta R_{\min}/\sqrt{2} \end{aligned} \tag{3.57}$$

as well as one of the corresponding equations with $n-1 \to n$. As shown in Appendix IV, these equations translate into the equations

$$\phi_{n-1} \in \begin{cases} \bigcup_{j=1}^{5} [A_{n-1,i}^{j,\mathrm{l}}, A_{n-1,i}^{j,\mathrm{u}}], & \cos\phi \geqslant 0 \\ \bigcup_{j=1}^{5} [\pi - A_{n-1,i}^{j,\mathrm{u}}, \pi - A_{n-1,i}^{j,\mathrm{l}}], & \cos\phi < 0 \end{cases} \tag{3.58}$$

while the corresponding set for $n-1 \to n$ gives a similar pair with $A_{n-1,i}^{j,\{\mathrm{l},\mathrm{u}\}} \to A_{n-1,i}^{j,\{\mathrm{l},\mathrm{u}\}}$. The definitions of the $A_{a,i}^{j\{\mathrm{l},\mathrm{u}\}}$ (which here are all shifted to lie in the interval $[-\pi/2, 3\pi/2]$), along with those of the booleans $\mathcal{N}_{a,i}^{j}$ and $\mathcal{S}_{a,i}^{j}$ used below, are given in Appendix IV. The intersection of the allowed sets for all i then gives the allowed region, within which we generate ϕ_{n-1} uniformly. The associated jacobian is again the number of allowed bins divided by the total number of bins. Thus, if we split each of the two regions for ϕ_{n-1} into $\hat{B}$ intervals and define

$$\begin{aligned} b_{\{n-1,n\},i}^{j,\{\mathrm{l},\mathrm{u}\}+} &= \left\lfloor \frac{(A_{\{n-1,n\},i}^{j,\{\mathrm{l},\mathrm{u}\}} - A_{\mathrm{l}})\hat{B}}{A_{\mathrm{u}} - A_{\mathrm{l}}} \right\rfloor \\ b_{\{n-1,n\},i}^{j,\{\mathrm{l},\mathrm{u}\}-} &= \left\lfloor \frac{(\pi - A_{\{n-1,n\},i}^{j,\{\mathrm{u},\mathrm{l}\}} - A_{\mathrm{l}})\hat{B}}{A_{\mathrm{u}} - A_{\mathrm{l}}} \right\rfloor. \end{aligned} \tag{3.59}$$

(Note the interchange of $\mathrm{u} \leftrightarrow \mathrm{l}$ in going from the As to the b^-s.) We start with all bins marked "allowed" ($A_\pm = \{\text{all bins}\}$) and iterate the following steps for all $i < n-1$ for which all the inequalities are non-trivial ($\bigwedge_{j=1}^{5} \mathcal{N}_{n-1,i}^{j} =$ "true"):

$$S_\pm := \varnothing;$$

$$\begin{aligned} k_f &= \lfloor (B_a^+ + B_a^-)\, x_{3n-4} \rfloor \\ \delta_f &= (B_a^+ + B_a^-)\, x_{3n-4} - k_f \end{aligned}$$

$$\left.\begin{aligned} b_f &= b_{k_f}^{+} \\ \phi_{n-1} &= A_{\mathrm{l}} + (A_{\mathrm{u}} - A_{\mathrm{l}}) \frac{b_f + \delta_f}{\hat{B}} \end{aligned}\right\}, \quad k_f < B_a^+ \tag{3.61}$$

$$\left.\begin{aligned} b_f &= b_{k_f - B_a^+}^{-} \\ \phi_{n-1} &= \pi - A_{\mathrm{l}} - (A_{\mathrm{u}} - A_{\mathrm{l}}) \frac{b_f + \delta_f}{\hat{B}} \end{aligned}\right\}, \quad k_f \geqslant B_a^+.$$

The jacobian associated with this angle is

$$J_{n-1}^{\phi} = \frac{A_{\mathrm{u}} - A_{\mathrm{l}}}{\pi} \frac{B_a^+ + B_a^-}{2\hat{B}}. \tag{3.62}$$

To obtain the overall weight factor, we must combine the jacobians of Eq. (3.26), (3.30), (3.38), (3.46), and (3.62) with the factors in the measure, Eq. (3.3), and the phase space weight for the final two particles; this yields a weight W,

$$W = \frac{\pi(2\pi)^{2-2n}}{2} J_{n-1}^{\theta} J_{n-1}^{\phi} \prod_{i=1}^{n-2} E_i J_i^E J_i^\theta J_i^\phi. \tag{3.63}$$

4. LIGHT PARTICLE PHASE SPACE

In this section, I generalize the constraints developed in the previous section to handle light but not massless particles. By "light" particle I mean a particle whose mass is smaller than the corresponding minimum energy or minimum transverse energy constraint. (Although some of the considerations in this section in principle apply to heavier particles as well, in practice it is not appropriate to apply cuts to these particles, since one is often interested in them but in practice it is more efficient to use a slightly weaker constraint, with

$$M_i^2 = \Bigg(\sum_{\substack{j=i+1 \\ m_j \geqslant m_{\text{pair}}/\sqrt{2}}}^{n} m_j \Bigg)^2 + \Bigg(\sum_{\substack{j=i+1 \\ m_j < m_{\text{pair}}/\sqrt{2}}}^{n} m_j^2 \Bigg) + \frac{n_<(n_< - 1)}{2} m_{\text{pair}}^2, \tag{4.3}$$

where $n_<$ is the number of particles after the current particle with masses less than $m_{\text{pair}}/\sqrt{2}$.

Define the dimensionless quantities

$$\begin{aligned} \lambda_i &= \frac{\sqrt{[P^2 - (M_i + m_i)^2][P^2 - (M_i - m_i)^2]}}{P^2} \\ \gamma_i &= \frac{P^2 - M_i^2 + m_i^2}{P^2} \\ \mu_i &= \frac{m_i}{P^0} \\ k_i &= \frac{|\mathbf{p}_i|}{P^0} \\ e_i &= \frac{p_i^0}{P^0} = \sqrt{k_i^2 + \mu_i^2}; \end{aligned} \tag{4.4}$$

Eq. (4.2) then leads to a maximum value for the energy

$$e_i \leqslant \frac{\gamma_i + \lambda_i v}{2} \tag{4.5}$$

(the corresponding limit on the norm of the momentum is $(\lambda_i + \gamma_i v)/2$) and then a constraint on $\cos \theta_{iP}$,

$$\cos \theta_{iP} \geqslant \frac{(v^2 - 1)\gamma_i + 2e_i}{2k_i v} \equiv L_{iP}^m. \tag{4.6}$$

The form of the constraints on θ_i and ϕ_i is then very similar to the one in the previous section; indeed, we need modify Eqs. (3.10) only by replacing L_{iP} with L_{iP}^m. So long as $\gamma_i \leqslant 1$ (which is usually true in practical applications), then $L_{iP}^m \geqslant L_{iP}$, and we can again retain the constraint of Eq. (3.17), as it will be weaker than (but still a reasonable approximation to) the corresponding constraint that would emerge from L_{iP}^m. (In the event that $L_{iP}^m < L_{iP}$, one would retain only the constraint $e_i \geqslant e_{\text{Tmin}}$.)

We may replace the longitudinal momentum constraint of Eq. (3.19) with a slightly weaker constraint on the longitudinal energy,

$$\max \sum_{j=i+1}^{n} |E_{jL}| \geqslant |P_L + p_{iL}|. \tag{4.7}$$

As shown in Appendix III, this leaves the additional bounds (3.21) in place.

The mass will, of course, cut off the soft divergences of the matrix elements, but if the mass is much smaller than the minimum energy cutoff, then the matrix element will be sharply peaked near the minimum energy, and it is still helpful to generate an extra factor of either the energy or the norm of the momentum to smooth out the integrand. The measure has the form

$$\text{const} \times |\mathbf{p}_i|^2 \frac{d\mathbf{p}_i}{\sqrt{|\mathbf{p}_i|^2 + m_i^2}} = \text{const} \times |\mathbf{p}_i| \, dE_i; \tag{4.8}$$

I leave a factor of $|\mathbf{p}_i| \, E_i$ explicit and generate the remainder through the mapping. For this purpose, one may again use Eq. (3.25); the jacobian (3.26) also carries over without change.

The various additional restrictions on $\cos \theta_i$ from Eqs. (3.13), (3.15), and (3.23) carry over without change, as do the generation of the polar angle, Eqs. (3.29)–(3.30), and the method of satisfying the ΔR constraint for the azimuthal angles, Eqs. (3.35)–(3.38).

For the final two particles, the measure in the light particle case is now (see Ref. [2])

$$\frac{1}{4} d\cos\theta_{n-1} \, d\phi_{n-1} \sum_{\pm} \frac{(k_{n-1}^{\pm})^2 \, \Theta(k_1^{\pm})}{k_{n-1}^{\pm} - v e_{n-1}^{\pm} \cos \theta_{n-1}}, \tag{4.9}$$

where

$$\begin{aligned} \rho &= 1 - v^2 + \mu_{n-1}^2 - \mu_n^2 = (1 - v^2)\gamma_{n-1} \\ k_{n-1}^{\pm} &= \frac{v\rho \cos\theta_{n-1} \pm \sqrt{\rho^2 - 4\mu_{n-1}^2 (1 - v^2 \cos^2 \theta_{n-1})}}{2(1 - v^2 \cos^2 \theta_{n-1})} \end{aligned} \tag{4.10}$$

Both solutions will contribute only in the case $k_{n-1}^- > 0$; this can arise only if $\rho < 2\mu_{n-1}$.

With $M_{n-1} = m_n$, we have the kinematic limits on e_{n-1},

$$\begin{aligned} e_{\text{l}}^{m\prime} &= \max\left(e_{\text{Tmin}}, \frac{\gamma_{n-1} - \lambda_{n-1} v}{2} \right) \\ e_{\text{u}}^{m\prime} &= \min\left(1 - e_{\text{Tmin}}, \frac{\gamma_{n-1} + \lambda_{n-1} v}{2} \right) \end{aligned} \tag{4.11}$$

which lead to corresponding limits on $\cos \theta_{n-1}$. (One should check explicitly that $e_{\text{l}}^{m\prime} < e_{\text{u}}^{m\prime}$, and the event should be rejected if this is not the case.) In addition, if $\rho < 2\mu_{n-1}$, there is an additional constraint on the cosine, since the particle can no longer travel in the direction opposite to $\mathbf{P}$. We should distinguish three cases, where the constraints are satisfied by (a) k^+_{n-1} alone, (b) k^-_{n-1} alone, and (c) both k^+_{n-1} and k^-_{n-1}. These three cases arise as follows,

$$\begin{array}{lll} k^+_{n-1} \text{ alone}, & \rho \geqslant \mu_{n-1} \text{ or } e_{\mathrm{l}}^{m\prime} \geqslant e^m_{\theta\max} \\ k^-_{n-1} \text{ alone}, & e_{\mathrm{u}}^{m\prime} < e^m_{\theta\max} \\ \text{both } k^+_{n-1} \text{ and } k^-_{n-1}, & \text{otherwise}, \end{array} \tag{4.12}$$

where

$$e^m_{\theta\max} = \frac{2\mu^2_{n-1}}{\rho}. \tag{4.13}$$

Define

$$k'_{\{\mathrm{l},\mathrm{u}\}} = \sqrt{(e^{m\prime}_{\{\mathrm{l},\mathrm{u}\}})^2 - \mu^2_{n-1}}. \tag{4.14}$$

In the first case, the limits on the cosine are

$$\begin{aligned} c^m_{\mathrm{l}} &= \max\left(-1, \frac{2e^{m\prime}_{\mathrm{l}} - \rho}{2vk'_{\mathrm{l}}}\right) \\ c^m_{\mathrm{u}} &= \min\left(1, \frac{2e^{m\prime}_{\mathrm{u}} - \rho}{2vk'_{\mathrm{u}}}\right), \end{aligned} \tag{4.15}$$

while in the second case, these are interchanged.

In the last case, we have a limit that demarcates the choice between k^+_{n-1} and k^-_{n-1},

$$c_{\mathrm{in}} = \frac{1}{v}\sqrt{1 - \rho^2/4\mu^2_{n-1}} \tag{4.16}$$

and a pair for the outer boundaries,

$$\begin{aligned} c^+_{\mathrm{out}} &= \min\left(1, \frac{2e^{m\prime}_{\mathrm{u}} - \rho}{2vk'_{\mathrm{u}}}\right) \\ c^-_{\mathrm{out}} &= \min\left(\frac{1, 2e^{m\prime}_{\mathrm{l}} - \rho}{2vk'_{\mathrm{l}}}\right). \end{aligned} \tag{4.17}$$

To generate the angle, one thus should set

$$\left.\begin{aligned} c^m_{\mathrm{l}} &= \max\left(-1, \frac{2e^{m\prime}_{\mathrm{l}} - \rho}{2vk'_{\mathrm{l}}}\right) \\ c^m_{\mathrm{u}} &= \min\left(1, \frac{2e^{m\prime}_{\mathrm{u}} - \rho}{2vk'_{\mathrm{u}}},\right) \\ \cos\theta_{n-1} &= (c^m_{\mathrm{u}} - c^m_{\mathrm{l}})\, x_{3n-5} + c^m_{\mathrm{l}} \\ k_{n-1} &= k^+_{n-1} \\ J^{m\theta}_{n-1} &= \frac{(c^m_{\mathrm{u}} - c^m_{\mathrm{l}})\, k^2_{n-1}}{k_{n-1} - ve_{n-1}\cos\theta_{n-1}} \end{aligned}\right\}, \quad \begin{array}{l} \rho \geqslant 2\mu_{n-1} \text{ or} \\ e^{m\prime}_{\mathrm{l}} \geqslant e^m_{\theta\max} \end{array}$$

$$\left.\begin{aligned} c^m_{\mathrm{l}} &= \min\left(1, \frac{2e^{m\prime}_{\mathrm{u}} - \rho}{2vk'_{\mathrm{u}}},\right) \\ c^m_{\mathrm{u}} &= \max\left(-1, \frac{2e^{m\prime}_{\mathrm{l}} - \rho}{2vk'_{\mathrm{l}}}\right) \\ \cos\theta_{n-1} &= (c^m_{\mathrm{u}} - c^m_{\mathrm{l}})\, x_{3n-5} + c^m_{\mathrm{l}} \\ k_{n-1} &= k^-_{n-1} \\ J^{m\theta}_{n-1} &= \frac{(c^m_{\mathrm{u}} - c^m_{\mathrm{l}})\, k^2_{n-1}}{k_{n-1} - ve_{n-1}\cos\theta_{n-1}} \end{aligned}\right\}, \quad e^{m\prime}_{\mathrm{u}} < e^m_{\theta\max} \tag{4.18}$$

$$\left.\begin{aligned} \hat{c}_{\mathrm{u}} &= c^{\mathrm{sign}(2x_{3n-5}-1)}_{\mathrm{out}} \\ \cos\theta_{n-1} &= |(\hat{c}_{\mathrm{u}} - c_{\mathrm{in}})(2x_{3n-5} - 1)| + c_{\mathrm{in}} \\ k_{n-1} &= k^{\mathrm{sign}(2x_{3n-5}-1)}_{n-1} \\ J^{m\theta}_{n-1} &= 2 \times \frac{(\hat{c}_{\mathrm{u}} - c_{\mathrm{in}})\, k^2_{n-1}}{|k_{n-1} - ve_{n-1}\cos\theta_{n-1}|} \end{aligned}\right\}, \quad \text{otherwise},$$

where c_{in} and $c^{\pm}_{\mathrm{out}}$ are given in Eqs. (4.16) and (4.17). (Once again, we have absorbed some of the factors from the measure into the jacobian.) These expressions for the jacobian $J^{m\theta}_{n-1}$ are unstable numerically if the denominator becomes small while k_{n-1} and $ve_{n-1}\cos\theta_{n-1}$ are not. This happens in practice occasionally when computing in single precision, either when μ_{n-1} is small and v is near 1, or in the last of the three cases considered above. In the latter case, one may replace the second expression for $J^{m\theta}_{n-1}$ in Eq. (4.18) with

$$D_{n-1} = 2v\mu_{n-1}\sqrt{|2x_{3n-5} - 1|(c_{\mathrm{in}} + \cos\theta_{n-1})}$$

$$J^{m\theta}_{n-1} = 2 \times \frac{\left(\begin{aligned} 4\sqrt{\hat{c}_{\mathrm{u}} - c_{\mathrm{in}}}\, k^2_{n-1}(1 - v^2\cos^2\theta_{n-1}) \\ \times (k_{n-1} + ve_{n-1}\cos\theta_{n-1}) \end{aligned}\right)}{\left(\begin{aligned} D_{n-1}\, |2v\rho\cos\theta_{n-1} + \mathrm{sign}(2x_{3n-5} - 1) \\ \times \sqrt{\hat{c}_{\mathrm{u}} - c_{\mathrm{in}}}\, D_{n-1}(1 + v^2\cos^2\theta_{n-1})| \end{aligned}\right)} \tag{4.19}$$

while in the former case, it is best to pick some small ε and an arbitrary $v_{\min} < 1$, and for $\mu_{n-1} < \varepsilon$, $v > v_{\min}$, replace both Eqs. (4.10) and (4.18) with

$$\begin{aligned} k_{n-1} &= \frac{e^{m\prime}_{\mathrm{u}} e^{m\prime}_{\mathrm{l}}}{e^{m\prime}_{\mathrm{u}} - (e^{m\prime}_{\mathrm{u}} - e^{m\prime}_{\mathrm{l}})\, x_{3n-5}} \\ J^{m\theta}_{n-1} &= \frac{(e^{m\prime}_{\mathrm{u}} - e^{m\prime}_{\mathrm{l}})\, k_{n-1}}{v(e^{m\prime}_{\mathrm{u}} - (e^{m\prime}_{\mathrm{u}} - e^{m\prime}_{\mathrm{l}})\, x_{3n-5})}. \end{aligned} \tag{4.20}$$

(These expressions are exact in the limit that $\varepsilon \to 0$.)

The energy fraction and cosine of the last particle are

$$\begin{aligned} e_n &= 1 - e_{n-1} \\ \cos\theta_n &= \frac{v - k_{n-1}\cos\theta_{n-1}}{k_n}. \end{aligned} \tag{4.21}$$

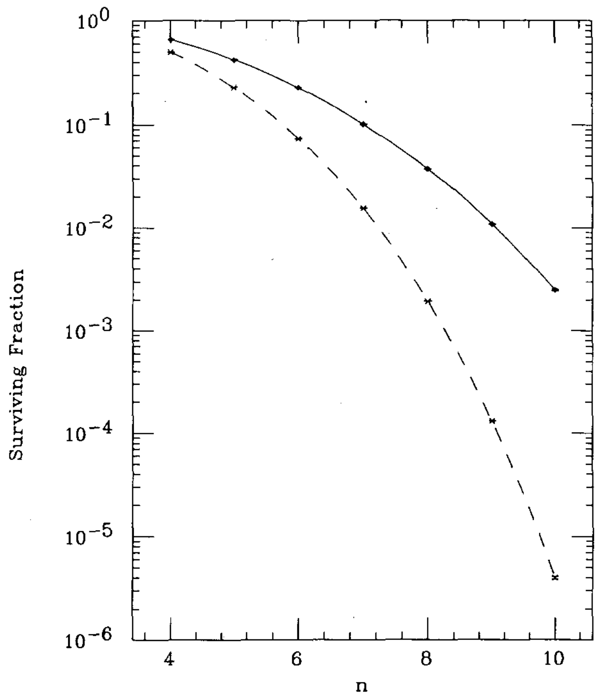

FIG. 1. The fractions of phase space surviving the cuts described in the text, as a function of the number of outgoing particles. The upper set of points corresponds to the first set of cuts ($E_{T\min} = 0.02\sqrt{s}$), while the lower set corresponds to the second set ($E_{T\min} = 0.05\sqrt{s}$).

Equations (3.50))(3.62), dealing with the azimuthal angle of the final pair, carry over to the present case without change. The final difference from the massless case comes in the formula for the weight, where Eq. (3.63) is replaced by a similar form,

$$W = \frac{\pi(2\pi)^{2-2n}}{2} J^{m\theta}_{n-1} J^{\phi}_{n-1} \prod_{i=1}^{n-2} |\mathbf{p}_i| \, J^E_i J^\theta_i J^\phi_i. \tag{4.22}$$

5. NUMERICAL EXAMPLES

As an example, I consider the integration of the function

$$A(p_a, p_b \to \{p_i\}_{i=1}^n) = \frac{s^2}{(a1)(12)(23)\cdots(n-1n)(nb)} \tag{5.1}$$

over n-particle phase space, where $(ij) = 2p_i \cdot p_j$; $s = (ab)$; and $P_{\text{tot}} = p_a + p_b = (\sqrt{s}, \mathbf{0})$. This function has the essential features of massless-particle amplitudes and soft and collinear singularities. Indeed, it is one of the terms in the non-vanishing Parke–Taylor helicity amplitude for multi-gluon scattering [7].

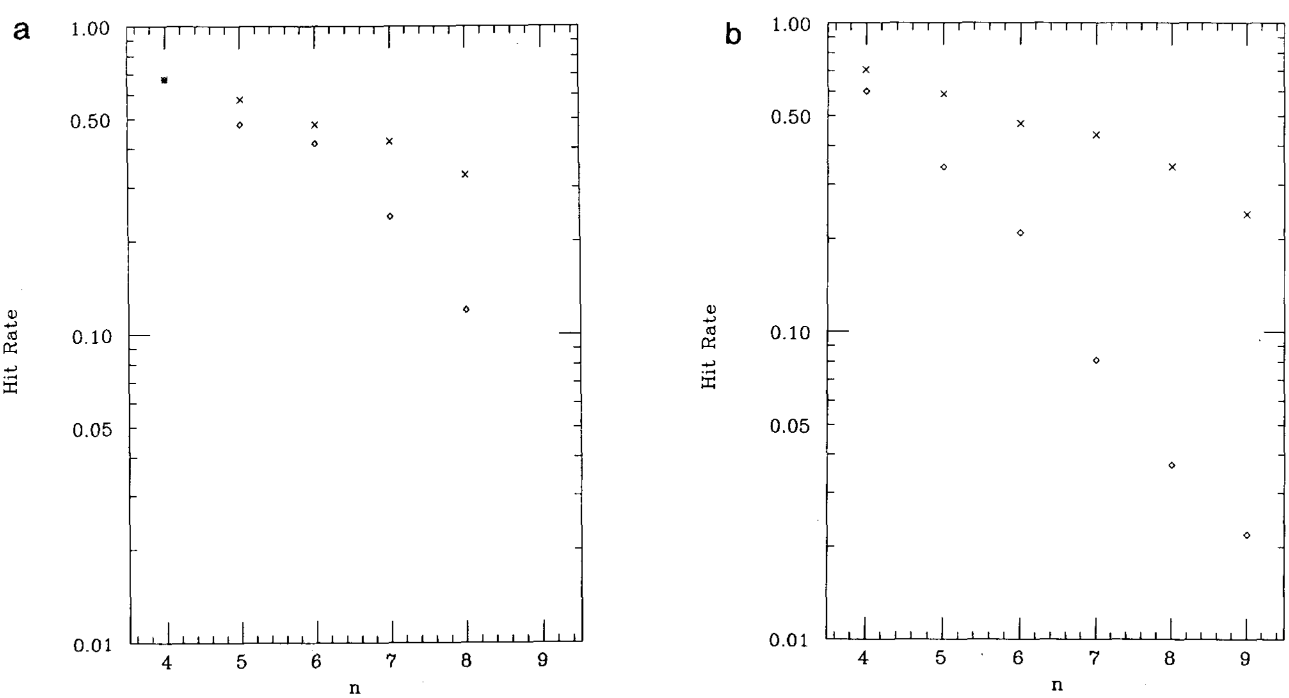

FIG. 2. The hit rates for the integral of Eq. (5.1). The hit rates for Rambo are plotted with the diamond symbol, those for Octopus with a cross: (a) $E_{T\min} = 0.02\sqrt{s}$; (b) $E_{T\min} = 0.05\sqrt{s}$.

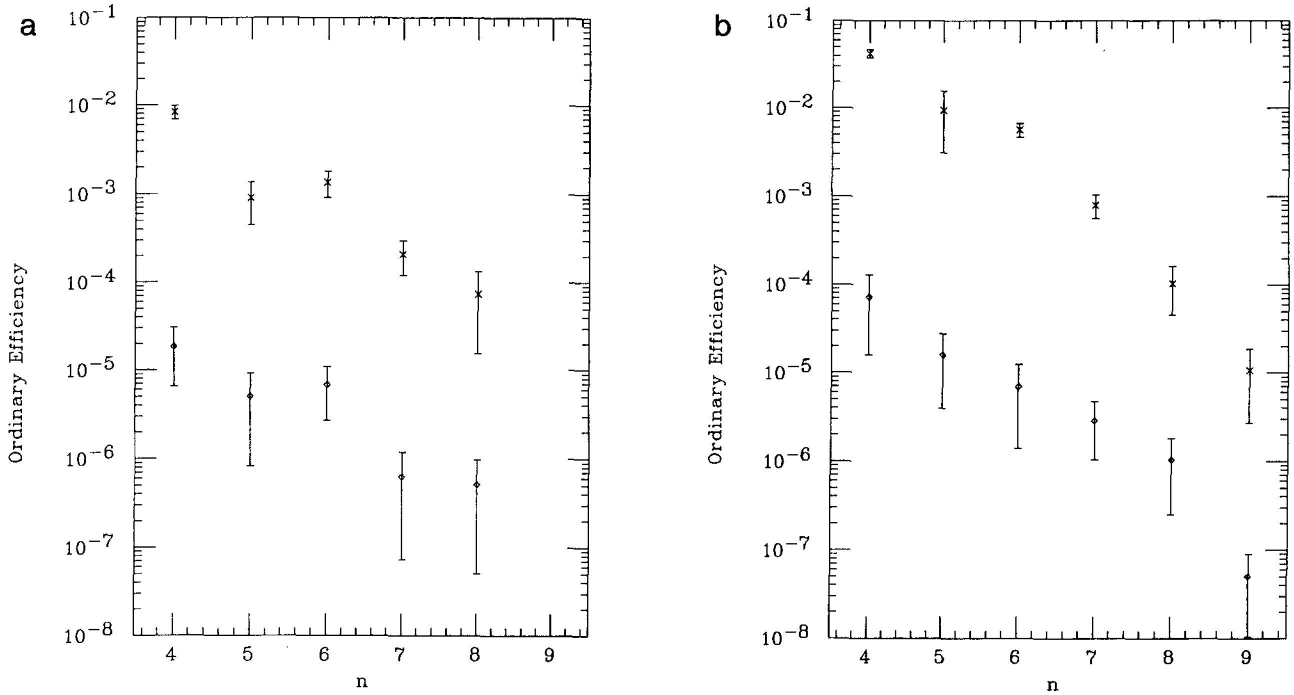

FIG. 3. The ordinary efficiencies for the integral of Eq. (5.1). The efficiencies for Rambo are plotted with the diamond symbol, those for Octopus with a cross: (a) $E_{T\min} = 0.02\sqrt{s}$; (b) $E_{T\min} = 0.05\sqrt{s}$.

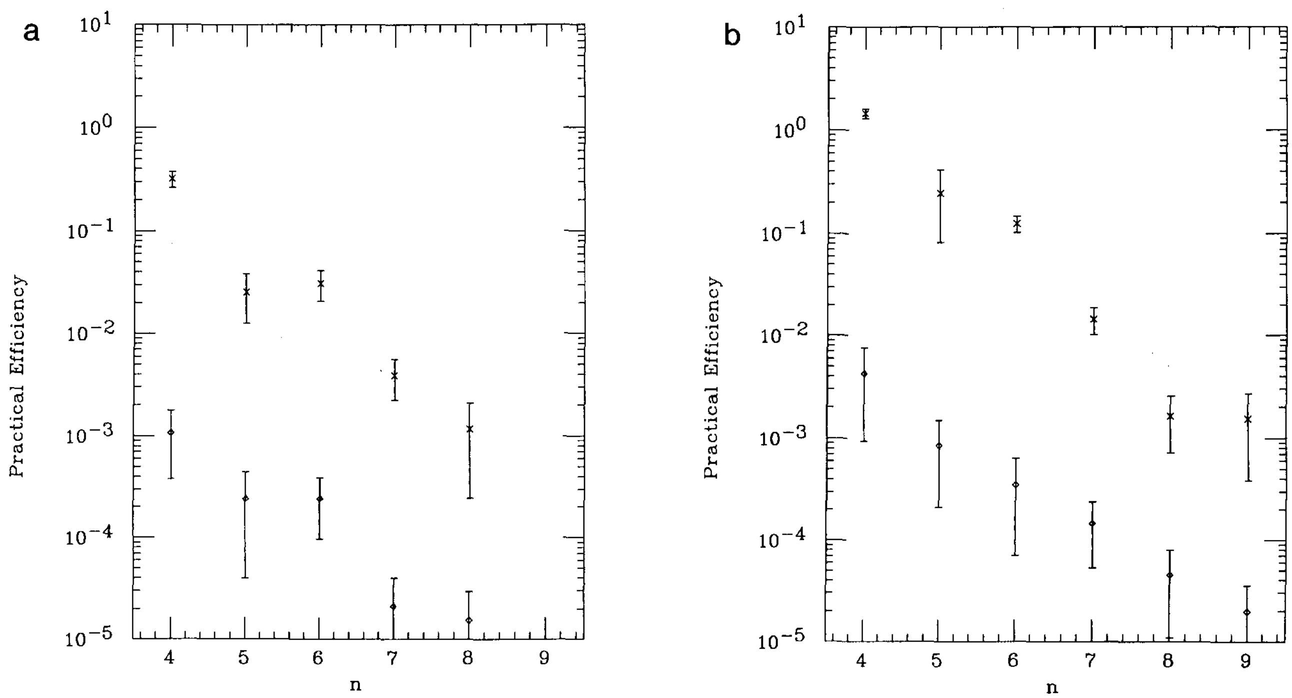

FIG. 4. The practical efficiencies for the integral of Eq. (5.1). The efficiencies for Rambo are plotted with the diamond symbol, those for Octopus with a cross: (a) $E_{T\min} = 0.02\sqrt{s}$; (b) $E_{T\min} = 0.05\sqrt{s}$.

I use two sample cuts:

(a) $E_{T\,\min} = 0.02\sqrt{s}$, $\Delta R_{\min} = 0.8$, $\eta_{\max} = 3.5$.

(b) $E_{T\,\min} = 0.05\sqrt{s}$, $\Delta R_{\min} = 0.8$, $\eta_{\max} = 3.5$.

In each calculation, Vegas was used to feed each of the phase space generators, Rambo and Octopus. Vegas was given five iterations (with a minimum of 2000 accepted events each) to refine its bins and adapt (as best it could) to the integrand. The Vegas grid was then frozen, and the run continued with sets of 10 iterations, increasing the number of points per iteration for each new set. These sets yield an estimate of the asymptotic efficiency of each of the generators in the particular calculation. In practice, I have simply chosen the error estimates corresponding to the iterations with the largest number of points. (One also must ensure that one has reached a regime where the estimates of the integral do not fluctuate too wildly, else Vegas's error estimates will usually be much too small.) The fractions of phase space surviving the two cuts are shown in Fig. 1; the scaling of the hit rate with the number of final state particles n for the two cuts is shown in Fig. 2; the scaling of the

trivially satisfied if $C_R < -1$, so we may assume that $C_R \in [-1, 1]$. Rewrite

$$\begin{aligned}\cos\theta_{i\mathbf{P}} &= \cos\theta_i\cos\theta_{\mathbf{P}} + \sin\theta_i\sin\theta_{\mathbf{P}}\cos(\phi_i - \phi_{\mathbf{P}})\\ &= \cos\theta_i\cos\theta_{\mathbf{P}}\\ &\quad + \sqrt{(1-\cos^2\theta_i)(1-\cos^2\theta_{\mathbf{P}})}\\ &\quad \times\cos(\phi_i - \phi_{\mathbf{P}}) \qquad (\mathrm{I.2})\end{aligned}$$

so that the constraint becomes

$$\cos(\phi_i - \phi_{\mathbf{P}}) \geqslant \frac{C_R - \cos\theta_i\cos\theta_{\mathbf{P}}}{\sqrt{(1-\cos^2\theta_i)(1-\cos^2\theta_{\mathbf{P}})}}. \qquad (\mathrm{I.3})$$

In order to allow a solution to this constraint, the right-hand side must be less than or equal to 1:

$$C_R - \cos\theta_i\cos\theta_{\mathbf{P}} \leqslant \sqrt{(1-\cos^2\theta_i)(1-\cos^2\theta_{\mathbf{P}})} \qquad (\mathrm{I.4})$$

Let us consider the first case in more detail. We can subdivide this into three sub-cases,

$$\begin{aligned} \frac{C_R}{\cos\theta_{\mathbf{P}}} > 1:&\quad \cos\theta_i \in [-1, 1] \\ \frac{C_R}{\cos\theta_{\mathbf{P}}} \in [C^-, C^+]:&\quad \cos\theta_i \in [-1, C^+] \\ \frac{C_R}{\cos\theta_{\mathbf{P}}} < -1:&\quad \cos\theta_i \in [C^-, C^+]. \end{aligned} \tag{I.9}$$

The function $x/\sqrt{1-x^2}$ is monotonically increasing for $x \in (-1, 1)$; thus,

$$\begin{aligned} \frac{C_R}{\cos\theta_{\mathbf{P}}} < C^- &\Rightarrow \frac{C_R}{\sqrt{1-C_R^2}} \\ &> -\frac{\cos\theta_{\mathbf{P}}}{\sqrt{1-\cos^2\theta_{\mathbf{P}}}} \Rightarrow \frac{C_R}{\cos\theta_{\mathbf{P}}} < -1, \\ \frac{C_R}{\cos\theta_{\mathbf{P}}} > C^+ &\Rightarrow \frac{C_R}{\sqrt{1-C_R^2}} \\ &< \frac{\cos\theta_{\mathbf{P}}}{\sqrt{1-\cos^2\theta_{\mathbf{P}}}} \Rightarrow \frac{C_R}{\cos\theta_{\mathbf{P}}} > 1, \end{aligned} \tag{I.10}$$

so the "missing" sub-cases in Eq. (I.9) are in fact forbidden. There is, of course, a similar sub-division in the case $\cos\theta_{\mathbf{P}} \geqslant 0$.

In summary, the constraints on θ_i are

$$\begin{aligned} \cos\theta_i &\in \left[\begin{Bmatrix} -1, & \dfrac{C_R}{\cos\theta_{\mathbf{P}}} > -1 \\ C^-, & \text{otherwise} \end{Bmatrix}, \begin{Bmatrix} 1, & \dfrac{C_R}{\cos\theta_{\mathbf{P}}} \geqslant 1 \\ C^+, & \text{otherwise} \end{Bmatrix}\right], \quad (\cos\theta_{\mathbf{P}} < 0) \\ \cos\theta_i &\in \left[\begin{Bmatrix} -1, & \dfrac{C_R}{\cos\theta_{\mathbf{P}}} \leqslant -1 \\ C^-, & \text{otherwise} \end{Bmatrix}, \begin{Bmatrix} 1, & \dfrac{C_R}{\cos\theta_{\mathbf{P}}} < 1 \\ C^+, & \text{otherwise} \end{Bmatrix}\right], \quad (\cos\theta_{\mathbf{P}} \geqslant 0). \end{aligned} \tag{I.11}$$

Upon shifting $\cos\theta_{\mathbf{P}}$ to the other side of inequalities and observing that, at the points where the two sides in the inequalities are equal, the two branches are also equal, then we can write them more simply as

$$\cos\theta_i \in \left[\begin{Bmatrix} -1, & C_R \leqslant -\cos\theta_{\mathbf{P}} \\ C^-, & \text{otherwise} \end{Bmatrix}, \begin{Bmatrix} 1, & C_R \leqslant \cos\theta_{\mathbf{P}} \\ C^+, & \text{otherwise} \end{Bmatrix}\right] \tag{I.12}$$

while the constraint on ϕ_i (given the above constraints on $\cos\theta_i$) may be written

$$|\phi_i - \phi_{\mathbf{P}}| \leqslant \operatorname{acos}\left(\frac{C_R - \cos\theta_i \cos\theta_{\mathbf{P}}}{\sqrt{(1-\cos^2\theta_i)(1-\cos^2\theta_{\mathbf{P}})}}\right), \tag{I.13}$$

where the range of acos is understood to be $[0, \pi]$ and where I adopt the convention that $\operatorname{acos}(x > 1) = 0$, $\operatorname{acos}(x < -1) = \pi$.

APPENDIX II: TRANSVERSE ENERGY CONSTRAINTS

We wish to discover what constraints on the energy are imposed by the requirement that the intersection of Eq. (3.10) and Eq. (3.15) be nontrivial, that is, by the pair of constraints

$$\begin{aligned} \sqrt{1-\left(\frac{e_{\mathrm{T\,min}}}{e_i}\right)^2} &> c^- \\ -\sqrt{1-\left(\frac{e_{\mathrm{T\,min}}}{e_i}\right)^2} &< c^+. \end{aligned} \tag{II.1}$$

In the case that $L_{iP} \leqslant -1$, the constraints are trivial, so we need consider only $L_{iP} \in [-1, 1]$, which implies that

$$-ve_i < e_i + \frac{v^2-1}{2} < ve_i \tag{II.2}$$

or

$$e_i \in \left[\frac{1-v}{2}, \frac{1+v}{2}\right]. \tag{II.3}$$

Of course, we must have $e_{\mathrm{T\,min}} < (1+v)/2$, else there is no range of allowed energies anyway.

I restrict attention to the case $\cos\theta_{\mathbf{P}} < 0$; the analysis for the other case is similar. If $L_{iP}/\cos\theta_{\mathbf{P}} > -1$, then the first constraint in Eq. (II.1) is trivially satisfied; whereas $L_{iP}/\cos\theta_{\mathbf{P}} \leqslant -1$ implies that $L_{iP} \geqslant -\cos\theta_{\mathbf{P}}$, which in turn implies that $L_{iP}^- < 0$. Since in this case, $c^- = L_{iP}^-$, the first constraint is again trivially satisfied.

The second constraint in Eq. (II.1) is satisfied trivially if $L_{iP}/\cos\theta_{\mathbf{P}} \geqslant 1$, so consider the remaining case $L_{iP}/\cos\theta_{\mathbf{P}} < 1$, or

$$e_i > \frac{1-v^2}{2(1-v\cos\theta_{\mathbf{P}})}. \tag{II.4}$$

If $L_{iP}^{+} > 0$, the constraint is again satisfied trivially. A non-trivial constraint will arise if $L_{iP}^{+} < 0$, or

$$\cos\theta_{\mathbf{P}} \frac{2e_i - 1 + v^2}{2v} + \sin\theta_{\mathbf{P}} \sqrt{e_i^2 - ((2e_i - 1 + v^2)/2v)^2 < 0)} \tag{II.5}$$

Introducing $\hat{x} = (2e_i - 1)/v$ (note that $\hat{x} \in [-1, 1]$), this condition becomes

$$\cos\theta_{\mathbf{P}}(\hat{x}+v) + \sin\theta_{\mathbf{P}} \sqrt{(1-v^2)(1-\hat{x}^2)} < 0 \tag{II.6}$$

or

$$x > -v \quad \text{and} \quad \cos^2\theta_{\mathbf{P}}(\hat{x}+v)^2 - \sin^2\theta_{\mathbf{P}}(1-v^2)(1-\hat{x}^2) > 0 \tag{II.7}$$

which tells us that

$$e_i > \frac{1-v^2}{2} \quad \text{and} \quad \hat{x} \notin [\hat{x}_-, \hat{x}_+], \tag{II.8}$$

where

$$\hat{x}_{\pm} = -\frac{\cos^2\theta_{\mathbf{P}} v \mp (1-v^2)\sin\theta_{\mathbf{P}}}{\cos^2\theta_{\mathbf{P}} + (1-v^2)\sin^2\theta_{\mathbf{P}}} = \frac{1-v^2}{2(1 \mp v\sin\theta_{\mathbf{P}})}. \tag{II.9}$$

With $e_{\pm} = (\hat{x}v + 1)/2$, we see that

$$e_+ \geqslant \frac{1-v^2}{2} \qquad e_- \leqslant \frac{1-v^2}{2}, \tag{II.10}$$

so this case is more simply $e_i \geqslant e_+$.

The second constraint of Eq. (II.1) becomes

$$-\sqrt{(\hat{x}v+1)^2 - 4e_{\mathrm{T}\min}^2} < \cos\theta_{\mathbf{P}}(\hat{x}+v) + \sin\theta_{\mathbf{P}} \sqrt{(1-v^2)(1-\hat{x}^2)}, \tag{II.11}$$

where the assumption $e_i > e_{\mathrm{T}\min}$ is implicit. Since both sides of the inequality are negative, it becomes

$$(\hat{x}v+1)^2 - 4e_{\mathrm{T}\min}^2 > \cos^2\theta_{\mathbf{P}}(\hat{x}+v)^2 + \sin^2\theta_{\mathbf{P}}(1-v^2)(1-\hat{x}^2) + 2\cos\theta_{\mathbf{P}}\sin\theta_{\mathbf{P}}(\hat{x}+v)\sqrt{(1-v^2)(1-\hat{x}^2)}. \tag{II.12}$$

In principle, the inequality can be solved exactly, but this involves the disgusting solutions to a quartic equation. We may, however, observe that

$$\begin{aligned} &+2\cos\theta_{\mathbf{P}}\sin\theta_{\mathbf{P}}(\hat{x}+v)\sqrt{(1-v^2)(1-\hat{x}^2)} \\ &\quad = -2\,|\cos\theta_{\mathbf{P}}|\,\sin\theta_{\mathbf{P}}(\hat{x}+v)\sqrt{(1-v^2)(1-\hat{x}^2)} \\ &\quad > -2\,|\cos\theta_{\mathbf{P}}|\,\sin\theta_{\mathbf{P}}(\hat{x}+v)\sqrt{(1-v^2)} \end{aligned} \tag{II.13}$$

and this gives a weaker (but simpler) constraint,

$$(\hat{x}v+1)^2 - 4e_{\mathrm{T}\min}^2 > \cos^2\theta_{\mathbf{P}}(\hat{x}+v)^2 + \sin^2\theta_{\mathbf{P}}(1-v^2)(1-\hat{x}^2) + 2\cos\theta_{\mathbf{P}}\sin\theta_{\mathbf{P}}(\hat{x}+v)\sqrt{(1-v^2)}. \tag{II.14}$$

The inequality has the solution

$$\begin{aligned} &\hat{x} \notin [\hat{z}_-, \hat{z}_+], && \alpha > 0 \\ &\hat{x} \in [\hat{z}_+, \hat{z}_-], && \alpha < 0, \end{aligned} \tag{II.15}$$

where

$$\hat{z}_{\pm} = \frac{\beta\sin\theta_{\mathbf{P}} \pm \sqrt{4\alpha e_{\mathrm{T}\min}^2 + \cos^2\theta_{\mathbf{P}}(1-v^2)\,\zeta^2}}{\alpha} \tag{II.16}$$

and where α and β are given in Equation (3.18),

$$\alpha = \sin^2\theta_{\mathbf{P}} - (1-v^2)\cos^2\theta_{\mathbf{P}}$$
$$\beta = \sqrt{1-v^2}\cos^2\theta_{\mathbf{P}} - v\sin\theta_{\mathbf{P}}.$$

If we define

$$y_{\pm} = \hat{z}_{\pm}|_{e_{\mathrm{T}\min}=0} = \beta\frac{\sin\theta_{\mathbf{P}} \pm \cos\theta_{\mathbf{P}}\sqrt{1-v^2}}{\alpha} = -\frac{v\sin\theta_{\mathbf{P}} - \sqrt{1-v^2}\cos\theta_{\mathbf{P}}}{\sin\theta_{\mathbf{P}} \mp \sqrt{1-v^2}\cos\theta_{\mathbf{P}}} \tag{II.17}$$

then the sign of the denominator of $\hat{y}_-$ is the same as the sign of α.

If $\alpha > 0$, then the discriminant δ inside the square root in Equation (II.16) is positive, and furthermore, $\hat{z}_- \leqslant \hat{y}_- \leqslant -1$, so $\hat{x}$ cannot be less than $\hat{z}$, and the constraint in Equation (II.15) reduces to $\hat{x} > \hat{z}_+$. If $\alpha < 0$ (which in practice happens less frequently), then if the discriminant is negative, the constraint cannot be solved, and we are left with the restriction that $e_i \leqslant e_+$. If the discriminant is positive, we may note that $\hat{z}_- \geqslant \hat{y}_- \geqslant 1$, so that the upper limit on $\hat{x}$ remains 1 (from the original kinematic limit), while the lower limit becomes $\hat{x} \geqslant \hat{z}_+$.

Putting all the constraints together, we obtain ($\delta > 0$)

$$e_i > e_{\mathrm{T\,min}} \qquad \text{and} \qquad e_i \notin [e_+, (v\hat{z}_+ + 1)/2]. \tag{II.18}$$

In practice, the region between $e_{\mathrm{T\,min}}$ and e_+ does not exist, and it is sufficient to consider these additional constraints only in the case that $e_{\mathrm{T\,min}} \geqslant e_+$ and $\delta > 0$.

APPENDIX III: TOTAL LONGITUDINAL MOMENTUM CONSTRAINTS

In the massless case, we want

$$\max \sum_{j=i+1}^{n} |p_{jL}| \geqslant |P_L - p_{iL}|. \tag{III.1}$$

The left-hand side can be re-expressed as

$$\sum_{j=i+1}^{n} \sqrt{E_j^2 - E_{\mathrm{T}j}^2} \tag{III.2}$$

which is maximized when $E_{\mathrm{T}j} = E_{\mathrm{T\,min}}$ for all the remaining particles. Furthermore,

$$\sqrt{E_j^2 - E_{\mathrm{T\,min}}^2} + \sqrt{E_l^2 - E_{\mathrm{T\,min}}^2} \geqslant \sqrt{(E_j + E_l - E_{\mathrm{T\,min}})^2 - E_{\mathrm{T\,min}}^2} \tag{III.3}$$

so the sum of absolute longitudinal momenta is maximized when the remaining energy is distributed equally amongst the momenta; this maximum is

$$\sqrt{(P^0 - E_i)^2 - (n-i)^2 E_{\mathrm{T\,min}}^2}. \tag{III.4}$$

In the case where different particles have different minimum transverse energies, the sum is maximized when the energy of each particle is proportional to its minimum transverse energy:

$$E_l \propto \frac{E_{\mathrm{T\,min}\,l}}{\sum_j E_{\mathrm{T\,min}\,j}}. \tag{III.5}$$

The maximum in this case has the value

$$\sqrt{(P^0 - E_i)^2 - \left(\sum_{j=i+1}^{n} E_{\mathrm{T\,min}\,j}\right)^2}. \tag{III.6}$$

With

$$s_{\mathrm{T}} = \frac{1}{P^0}\left(\sum_{j=i+1}^{n} E_{\mathrm{T\,min}\,j}\right),$$

we thus have the constraint

$$\big|\, |\mathbf{P}| \cos\theta_{\mathbf{P}} - E_i \cos\theta_i \big| \leqslant \sqrt{(P^0 - E_i)^2 - (P^0)^2 s_{\mathrm{T}}^2}. \tag{III.7}$$

The left-hand side has its minimum when

$$\cos\theta_i = \mathrm{sign}(\cos\theta_{\mathbf{P}}) \times \min(|\cos\theta_{\mathbf{P}}|\, |\mathbf{P}|/E_i, \sqrt{1 - e_{\mathrm{T\,min}}^2/e_i^2}); \tag{III.8}$$

in order to allow a solution at all, the value there must be less than the right-hand side. If $e_i \geqslant \sqrt{e_{\mathrm{T\,min}}^2 + v^2 \cos^2\theta_{\mathbf{P}}}$, the minimum of the left-hand side is zero, and there is no constraint; otherwise we must have

$$\big|v\, |\cos\theta_{\mathbf{P}}| - e_i \sqrt{1 - e_{\mathrm{T\,min}}^2/e_i^2}\big| \leqslant \sqrt{(1 - e_i)^2 - s_{\mathrm{T}}^2}; \tag{III.9}$$

squaring both sides we obtain

$$-2v\, |\cos\theta_{\mathbf{P}}| \sqrt{e_i^2 - e_{\mathrm{T\,min}}^2} \leqslant e_{\mathrm{T\,min}}^2 + 1 - v^2 \cos^2\theta_{\mathbf{P}} - s_{\mathrm{T}}^2 - 2e_i \tag{III.10}$$

which means that

$$e_i \leqslant \omega/2 \qquad \text{or} \qquad e_i \in [e_-, e_+], \tag{III.11}$$

where

$$\begin{aligned} \omega &= 1 - v^2 \cos^2\theta_{\mathbf{P}} - s_{\mathrm{T}}^2 + e_{\mathrm{T\,min}}^2 \\ \chi &= 1 - \frac{e_{\mathrm{T\,min}}^2 (1 - v^2 \cos^2\theta_{\mathbf{P}})}{\omega^2} \\ e_\pm &= \frac{\omega}{2(1 - v^2\cos^2\theta_{\mathbf{P}})} (1 \pm v\, |\cos\theta_{\mathbf{P}}| \sqrt{\chi}). \end{aligned} \tag{III.12}$$

Combining the two, we obtain

$$\begin{aligned} e_i &\leqslant e_+, \qquad \chi > 0 \\ e_i &\leqslant \omega/2, \qquad \chi < 0 \end{aligned} \tag{III.13}$$

so long as either $\chi < 0$, $e_- \leqslant \omega/2$, or $e_- < e_{\min}$, which is always true in practice.

Equation (III.7) then yields the following constraint on $\cos\theta_i$;

$$\cos\theta_i \in \left[\frac{v\cos\theta_{\mathrm{P}} - \sqrt{(1-e_i)^2 - s_{\mathrm{T}}^2}}{e_i}, \frac{v\cos\theta_{\mathrm{P}} + \sqrt{(1-e_i)^2 - s_{\mathrm{T}}^2}}{e_i}\right]. \tag{III.14}$$

In the massive case, we replace the constraint (III.1) with the slightly weaker constraint

$$\max \sum_{j=i+1}^{n} |E_{jL}| \geqslant |P_L + p_{iL}|. \tag{III.15}$$

The right-hand side of the inequality (III.7) is then unchanged. In the left-hand side, e_i should be replaced by k_i; but in the case that $e_i < \sqrt{e_{\mathrm{T\,min}}^2 + v^2\cos^2\theta_{\mathrm{P}}}$, leaving the e_i in place gives a weaker (but safe) constraint.

APPENDIX IV: CONSTRAINTS ON THE ANGLES OF PENULTIMATE AND ULTIMATE MOMENTA

We wish to simplify the set of inequalities

$$\operatorname{atanh}\cos\theta_{n-1}^{L} < \eta_i - \Delta R_{\min}/\sqrt{2} \tag{IV.1a}$$

$$\operatorname{atanh}\cos\theta_{n-1}^{L} > \eta_i + \Delta R_{\min}/\sqrt{2} \tag{IV.1b}$$

$$\phi_{n-1}^{L} - \phi_i > \Delta R_{\min}/\sqrt{2} \tag{IV.1c}$$

$$\phi_{n-1}^{L} - \phi_i < -\Delta R_{\min}/\sqrt{2}, \tag{IV.1d}$$

where

$$\begin{aligned}
\cos\theta_{n-1}^{L} &= \cos\theta_{n-1}\cos\theta_{\mathrm{P}} \\
&\quad + \sin\theta_{n-1}\sin\theta_{\mathrm{P}}\sin\phi_{n-1} \\
\sin\theta_{n-1}^{L}\sin\phi_{n-1}^{L} &= \cos\theta_{n-1}\sin\theta_{\mathrm{P}}\sin\phi_{\mathrm{P}} \\
&\quad + \sin\theta_{n-1}\cos\phi_{n-1}\cos\theta_{\mathrm{P}} \\
&\quad - \sin\theta_{n-1}\sin\phi_{n-1}\cos\theta_{\mathrm{P}}\sin\phi_{\mathrm{P}} \\
\sin\theta_{n-1}^{L}\cos\phi_{n-1}^{L} &= \cos\theta_{n-1}\sin\theta_{\mathrm{P}}\cos\phi_{\mathrm{P}} \\
&\quad - \sin\theta_{n-1}\cos\phi_{n-1}\sin\theta_{\mathrm{P}} \\
&\quad - \sin\theta_{n-1}\sin\phi_{n-1}\cos\theta_{\mathrm{P}}\cos\phi_{\mathrm{P}}.
\end{aligned} \tag{IV.2}$$

Recall that we are keeping track of two separate regions for ϕ_{n-1}, one where $\cos\phi_{n-1} \geqslant 0$, the other where the cosine is negative. Let us restrict attention for a while to the first region. Inequalities (IV.1a), (IV.1b) are the easiest; taking the hyperbolic tangent of both sides and using the previous equation, we obtain

$$\begin{aligned}
\sin\phi_{n-1} &< \frac{1}{\sin\theta_{\mathrm{P}}\sin\theta_{n-1}} \\
&\quad \times \left(\frac{\cos\theta_i - t_{\min}}{1 - t_{\min}\cos\theta_i} - \cos\theta_{\mathrm{P}}\cos\theta_{n-1}\right) \\
&\equiv s_{n-1,i}^{1} \\
\sin\phi_{n-1} &> \frac{1}{\sin\theta_{\mathrm{P}}\sin\theta_{n-1}} \\
&\quad \times \left(\frac{\cos\theta_i + t_{\min}}{1 + t_{\min}\cos\theta_i} - \cos\theta_{\mathrm{P}}\cos\theta_{n-1}\right) \\
&\equiv s_{n-1,i}^{2},
\end{aligned} \tag{IV.3}$$

where $t_{\min} = \tanh(\Delta R_{\min}/\sqrt{2})$. With the range of asin understood to be $[-\pi/2, \pi/2]$, and

$$\begin{aligned}
|s_{n-1,i}^{1}| \leqslant 1:&\quad \begin{cases} A_{n-1,i}^{1,\mathrm{u}} = \operatorname{asin} s_{n-1,i}^{1} \\ A_{n-1,i}^{1,1} = \pi - A_{n-1,i}^{1,u} \\ \mathscr{S}_{n-1,i}^{1} = \text{“true”}, \\ \mathscr{N}_{n-1,i}^{1} = \text{“true”} \end{cases} \\
s_{n-1,i}^{1} > 1:&\quad \mathscr{N}_{n-i,i}^{1} = \text{“false”} \\
s_{n-1,i}^{1} < -1:&\quad \begin{cases} \mathscr{S}_{n-1,i}^{1} = \text{“false”}, \\ \mathscr{N}_{n-1,i}^{1} = \text{“true”} \end{cases} \\
|s_{n-1,i}^{2}| \leqslant 1:&\quad \begin{cases} A_{n-1,i}^{2,1} = \operatorname{asin} s_{n-1,i}^{2} \\ A_{n-1,i}^{2,\mathrm{u}} = \pi - A_{n-1,i}^{2,l} \\ \mathscr{S}_{n-1,i}^{2} = \text{“true”}, \\ \mathscr{N}_{n-1,i}^{2} = \text{“true”} \end{cases} \\
s_{n-1,i}^{2} > 1:&\quad \begin{cases} \mathscr{S}_{n-1,i}^{2} = \text{“false”}, \\ \mathscr{N}_{n-1,i}^{2} = \text{“true”} \end{cases} \\
s_{n-1,i}^{2} < -1:&\quad \mathscr{N}_{n-1,i}^{2} = \text{“false”},
\end{aligned} \tag{IV.4}$$

these first inequalities become (in the non-trivial case with a solution)

$$\phi_{n-1} \in ([A_{n-1,i}^{1,1}, A_{n-1,i}^{1,u}] \cup [A_{n-1,i}^{2,1}, A_{n-1,i}^{2,\mathrm{u}}]) \bmod 2\pi. \tag{IV.5}$$

(It may be desirable to replace the asin function in this equation with a computationally cheaper approximation; this will require a shifting of the bins in Section 3, to account for the maximal possible error.)

The remaining two inequalities in Eq. (IV.1) we may replace, by the following trio,

$$\begin{gathered}
\sin(\phi_{n-1}^{L} - (\phi_i + \Delta R_{\min}/\sqrt{2})) > 0 \\
\sin(\phi_{n-1}^{L} - (\phi_i - \Delta R_{\min}/\sqrt{2})) < 0 \\
\cos(\phi_{n-1}^{L} - \phi_i) \leqslant 0,
\end{gathered} \tag{IV.6}$$

where the allowed region will consist of those ϕ_{n-1} which satisfy any one of the constraints.

Introduce

$$
\begin{aligned}
c_{+\Delta} &= \cos(\phi_i - \phi_{\mathbf{P}} + \Delta R_{\min}/\sqrt{2}),\\
c_{-\Delta} &= \cos(\phi_i - \phi_{\mathbf{P}} - \Delta R_{\min}/\sqrt{2}),\\
c_0 &= \cos(\phi_i - \phi_{\mathbf{P}}),\\
s_{+\Delta} &= \sin(\phi_i - \phi_{\mathbf{P}} + \Delta R_{\min}/\sqrt{2})\\
s_{-\Delta} &= \sin(\phi_i - \phi_{\mathbf{P}} - \Delta R_{\min}/\sqrt{2})\\
s_0 &= \sin(\phi_i - \phi_{\mathbf{P}}).
\end{aligned}
\tag{IV.7}
$$

Expanding the trigonometric functions, and using Equation (IV.2), we can rewrite these inequalities as

$$
\begin{aligned}
&-s_{+\Delta} \cos\theta_{n-1} \sin\theta_{\mathbf{P}} + \sin\theta_{n-1}\\
&\qquad \times (\cos\theta_{\mathbf{P}} s_{+\Delta} \sin\phi_{n-1} + c_{+\Delta}\cos\phi_{n-1}) > 0\\
&-s_{-\Delta} \cos\theta_{n-1} \sin\theta_{\mathbf{P}} + \sin\theta_{n-1}\\
&\qquad \times (\cos\theta_{\mathbf{P}}\, s_{-\Delta} \sin\phi_{n-1} + c_{-\Delta}\cos\phi_{n-1}) < 0\\
&c_0 \cos\theta_{n-1} \sin\theta_{\mathbf{P}} - \sin\theta_{n-1}\\
&\qquad \times (\cos\theta_{\mathbf{P}} c_0 \sin\phi_{n-1} - s_0 \cos\phi_{n-1}) \leqslant 0.
\end{aligned}
\tag{IV.8}
$$

Define

$$
\begin{aligned}
n_{\pm\Delta} &= \sqrt{\cos^2\theta_{\mathbf{P}} s_{\pm\Delta}^2 + c_{\pm\Delta}^2}\\
n_0 &= \sqrt{\cos^2\theta_{\mathbf{P}} c_0^2 + s_0^2}\\
s_{n-1,i}^3 &= \frac{s_{+\Delta}\cos\theta_{n-1}\sin\theta_{\mathbf{P}}}{\sin\theta_{n-1} n_{+\Delta}}\\
s_{n-1,i}^4 &= \frac{s_{-\Delta}\cos\theta_{n-1}\sin\theta_{\mathbf{P}}}{\sin\theta_{n-1} n_{-\Delta}}\\
s_{n-1,i}^5 &= \frac{c_0\cos\theta_{n-1}\sin\theta_{\mathbf{P}}}{\sin\theta_{n-1}\, n_0}
\end{aligned}
\tag{IV.9}
$$

and then

$|s_{n-1,i}^3| \leqslant 1$:

$$
\left\{
\begin{array}{l}
\left.\begin{array}{l}
A_{n-1,i}^{3,\mathrm{l}} = \operatorname{asin}[s_{n-1,i}^3] - \operatorname{asin}\left[\dfrac{c_{+\Delta}}{n_{+\Delta}}\right]\\
A_{n-1,i}^{3,\mathrm{u}} = \pi - \operatorname{asin}[s_{n-1,i}^3] - \operatorname{asin}\left[\dfrac{c_{+\Delta}}{n_{+\Delta}}\right]
\end{array}\right\}, \quad s_{+\Delta}\cos\theta_{\mathbf{P}} \geqslant 0\\
\left.\begin{array}{l}
A_{n-1,i}^{3,\mathrm{l}} = \pi + \operatorname{asin}[s_{n-1,i}^3] + \operatorname{asin}\left[\dfrac{c_{+\Delta}}{n_{+\Delta}}\right]\\
A_{n-1,i}^{3,\mathrm{u}} = -\operatorname{asin}[s_{n-1,i}^3] + \operatorname{asin}\left[\dfrac{c_{+\Delta}}{n_{+\Delta}}\right]
\end{array}\right\}, \quad s_{+\Delta}\cos\theta_{\mathbf{P}} < 0\\
\mathscr{S}_{n-1,i}^3 = \text{“true”},\\
\mathscr{N}_{n-1,i}^3 = \text{“true”}
\end{array}
\right.
$$

$$
\begin{array}{ll}
s_{n-1,i}^3 > 1: & \mathscr{S}_{n-1,i}^3 = \text{“false”}, \quad \mathscr{N}_{n-1,i}^3 = \text{“true”}\\
s_{n-1,i}^3 < -1: & \mathscr{N}_{n-1,i}^3 = \text{“false”};
\end{array}
$$

$|s_{n-1,i}^4| \leqslant 1$:

$$
\left\{
\begin{array}{l}
\left.\begin{array}{l}
A_{n-1,i}^{4,\mathrm{l}} = \pi - \operatorname{asin}[s_{n-1,i}^4] - \operatorname{asin}\left[\dfrac{c_{-\Delta}}{n_{-\Delta}}\right]\\
A_{n-1,i}^{4,\mathrm{u}} = \operatorname{asin}[s_{n-1,i}^4] - \operatorname{asin}\left[\dfrac{c_{-\Delta}}{n_{-\Delta}}\right]
\end{array}\right\}, \quad s_{-\Delta}\cos\theta_{\mathbf{P}} \geqslant 0\\
\left.\begin{array}{l}
A_{n-1,i}^{4,\mathrm{l}} = -\operatorname{asin}[s_{n-1,i}^4] + \operatorname{asin}\left[\dfrac{c_{-\Delta}}{n_{-\Delta}}\right]\\
A_{n-1,i}^{4,\mathrm{u}} = \pi + \operatorname{asin}[s_{n-1,i}^4] + \operatorname{asin}\left[\dfrac{c_{-\Delta}}{n_{-\Delta}}\right]
\end{array}\right\}, \quad s_{-\Delta}\cos\theta_{\mathbf{P}} < 0\\
\mathscr{S}_{n-1,i}^4 = \text{“true”},\\
\mathscr{N}_{n-1,i}^4 = \text{“true”}
\end{array}
\right.
$$

$$
\begin{array}{ll}
s_{n-1,i}^4 > 1: & \mathscr{N}_{n-1,i}^4 = \text{“false”}\\
s_{n-1,i}^4 < -1: & \mathscr{S}_{n-1,i}^4 = \text{“false”}, \quad \mathscr{N}_{n-1,i}^4 = \text{“true”};
\end{array}
$$

and

$|s_{n-1,i}^5| \leqslant 1$:

$$
\left\{
\begin{array}{l}
\left.\begin{array}{l}
A_{n-1,i}^{5,\mathrm{l}} = \operatorname{asin}[s_{n-1,i}^5] + \operatorname{asin}\left[\dfrac{s_0}{n_{+\Delta}}\right]\\
A_{n-1,i}^{5,\mathrm{u}} = \pi - \operatorname{asin}[s_{n-1,i}^5] + \operatorname{asin}\left[\dfrac{s_0}{n_{+\Delta}}\right]
\end{array}\right\}, \quad c_0\cos\theta_{\mathbf{P}} \geqslant 0\\
\left.\begin{array}{l}
A_{n-1,i}^{5,\mathrm{l}} = \pi + \operatorname{asin}[s_{n-1,i}^5] - \operatorname{asin}\left[\dfrac{s_0}{n_{+\Delta}}\right]\\
A_{n-1,i}^{5,\mathrm{u}} = -\operatorname{asin}[s_{n-1,i}^5] - \operatorname{asin}\left[\dfrac{s_0}{n_{+\Delta}}\right]
\end{array}\right\}, \quad c_0\cos\theta_{\mathbf{P}} < 0\\
\mathscr{S}_{n-1,i}^5 = \text{“true”},\\
\mathscr{N}_{n-1,i}^5 = \text{“true”}
\end{array}
\right.
$$

$$
\begin{array}{ll}
s_{n-1,i}^5 > 1: & \mathscr{S}_{n-1,i}^5 = \text{“false”}, \quad \mathscr{N}_{n-1,i}^5 = \text{“true”}\\
s_{n-1,i}^5 < -1: & \mathscr{N}_{n-1,i}^5 = \text{“false.”}
\end{array}
\tag{IV.10}
$$

The inequalities of Eq. (IV.8) then have the solutions

$$\phi_{n-1} \in \begin{cases} \bigcup_{j=3}^{5} [A^{j,\mathrm{l}}_{n-1,i}, A^{j,\mathrm{u}}_{n-1,i}], & \cos\phi \geqslant 0 \\ \bigcup_{j=3}^{5} [\pi - A^{j,\mathrm{u}}_{n-1,i}, \pi - A^{j,\mathrm{l}}_{n-1,i}], & \cos\phi < 0. \end{cases} \tag{IV.11}$$

(This same form also allows us to include the inequalities (IV.1a), (IV.1b).)

The corresponding inequalities for $\sin\phi_n$ will also yield constraints on ϕ_{n-1}, since $\phi_{n-1} = \pi + \phi_n$. Define a set of $s^j_{n,i}$ in a similar manner to the definitions of the $s^j_{n-1,i}$ in Eq. (IV.3) and (IV.9). There then are a variety of minus signs and exchanges between lower and upper bounds that make a difference, but otherwise, the definitions of the $A^{j,\{\mathrm{l,u}\}}_{n,i}$ parallel those for $A^{j,\{\mathrm{l,u}\}}_{n-1,i}$:

$$|s^1_{n,i}| \leqslant 1: \quad \begin{cases} A^{1,\mathrm{l}}_{n,i} = \operatorname{asin} s^1_{n,i} \\ A^{1,\mathrm{u}}_{n,i} = \pi - A^{1,\mathrm{l}}_{n,i} \\ \mathscr{S}^1_{n,i} = \text{“true”}, \quad \mathscr{N}^1_{n,i} = \text{“true”} \end{cases}$$

$$s^1_{n,i} > 1: \quad \mathscr{N}^1_{n,i} = \text{“false”}$$

$$s^1_{n,i} < -1: \quad \mathscr{S}^1_{n,i} = \text{“false,”} \quad \mathscr{N}^1_{n,i} = \text{“true”} \tag{IV.12}$$

$$|s^2_{n,i}| \leqslant 1: \quad \begin{cases} A^{2,\mathrm{u}}_{n,i} = \operatorname{asin} s^2_{n,i} \\ A^{2,\mathrm{l}}_{n,i} = \pi - A^{2,\mathrm{u}}_{n,i} \\ \mathscr{S}^2_{n,i} = \text{“true,”} \quad \mathscr{N}^2_{n,i} = \text{“true”} \end{cases}$$

$$s^2_{n,i} > 1: \quad \mathscr{S}^2_{n,i} = \text{“false,”} \quad \mathscr{N}^2_{n,i} = \text{“true”}$$

$$s^2_{n,i} < -1: \quad \mathscr{N}^2_{n,i} = \text{“false”};$$

$$|s^3_{n,i}| \leqslant 1: \quad \begin{cases} \left.\begin{aligned} A^{3,\mathrm{l}}_{n,i} &= \pi + \operatorname{asin}[s^3_{n,i}] - \operatorname{asin}\left[\frac{c_{+\Delta}}{n_{+\Delta}}\right] \\ A^{3,\mathrm{u}}_{n,i} &= -\operatorname{asin}[s^3_{n,i}] - \operatorname{asin}\left[\frac{c_{+\Delta}}{n_{+\Delta}}\right] \end{aligned}\right\}, & s_{+\Delta}\cos\theta_{\mathbf{P}} \geqslant 0 \\ \left.\begin{aligned} A^{3,\mathrm{l}}_{n,i} &= \operatorname{asin}[s^3_{n,i}] + \operatorname{asin}\left[\frac{c_{+\Delta}}{n_{+\Delta}}\right] \\ A^{3,\mathrm{u}}_{n,i} &= \pi - \operatorname{asin}[s^3_{n,i}] + \operatorname{asin}\left[\frac{c_{+\Delta}}{n_{+\Delta}}\right] \end{aligned}\right\}, & s_{+\Delta}\cos\theta_{\mathbf{P}} < 0 \\ \mathscr{S}^3_{n,i} = \text{“true,”} \\ \mathscr{N}^3_{n,i} = \text{“true”} \end{cases}$$

$$s^3_{n,i} > 1: \quad \mathscr{S}^3_{n,i} = \text{“false,”} \quad \mathscr{N}^3_{n,i} = \text{“true”}$$

$$s_{n,i} < -1: \quad \mathscr{N}^3_{n,i} = \text{“false”};$$

$$|s^4_{n,i}| \leqslant 1: \quad \begin{cases} \left.\begin{aligned} A^{4,\mathrm{l}}_{n,i} &= -\operatorname{asin}[s^4_{n,i}] - \operatorname{asin}\left[\frac{c_{-\Delta}}{n_{-\Delta}}\right] \\ A^{4,\mathrm{u}}_{n,i} &= \pi + \operatorname{asin}[s^4_{n,i}] - \operatorname{asin}\left[\frac{c_{-\Delta}}{n_{-\Delta}}\right] \end{aligned}\right\}, & s_{-\Delta}\cos\theta_{\mathbf{P}} \geqslant 0 \\ \left.\begin{aligned} A^{4,\mathrm{l}}_{n,i} &= \pi - \operatorname{asin}[s^4_{n,i}] + \operatorname{asin}\left[\frac{c_{-\Delta}}{n_{-\Delta}}\right] \\ A^{4,\mathrm{u}}_{n,i} &= \operatorname{asin}[s^4_{n,i}] + \operatorname{asin}\left[\frac{c_{-\Delta}}{n_{-\Delta}}\right] \end{aligned}\right\}, & s_{-\Delta}\cos\theta_{\mathbf{P}} < 0 \\ \mathscr{S}^4_{n,i} = \text{“true,”} \\ \mathscr{N}^4_{n,i} = \text{“true”} \end{cases} \tag{IV.13}$$

$$s^4_{n,i} > 1: \quad \mathscr{N}^4_{n,i} = \text{“false”}$$

$$s^4_{n,i} < -1: \quad \mathscr{S}^4_{n,i} = \text{“false,”} \quad \mathscr{N}^4_{n,i} = \text{“true”};$$

$$|s^5_{n,i}| \leqslant 1: \quad \begin{cases} \left.\begin{aligned} A^{5,\mathrm{l}}_{n,i} &= \pi + \operatorname{asin}[s^5_{n,i}] + \operatorname{asin}\left[\frac{s_0}{n_{+\Delta}}\right] \\ A^{5,\mathrm{u}}_{n,i} &= -\operatorname{asin}[s^5_{n,i}] + \operatorname{asin}\left[\frac{s_0}{n_{+\Delta}}\right] \end{aligned}\right\}, & c_0\cos\theta_{\mathbf{P}} \geqslant 0 \\ \left.\begin{aligned} A^{5,\mathrm{l}}_{n,i} &= \operatorname{asin}[s^5_{n,i}] - \operatorname{asin}\left[\frac{s_0}{n_{+\Delta}}\right] \\ A^{5,\mathrm{u}}_{n,i} &= \pi - \operatorname{asin}[s^5_{n,i}] - \operatorname{asin}\left[\frac{s_0}{n_{+\Delta}}\right] \end{aligned}\right\}, & c_0\cos\theta_{\mathbf{P}} < 0 \\ \mathscr{S}^5_{n,i} = \text{“true,”} \\ \mathscr{N}^5_{n,i} = \text{“true”} \end{cases}$$

$$s^5_{n,i} > 1: \quad \mathscr{S}^5_{n,i} = \text{“false,”} \quad \mathscr{N}^5_{n,i} = \text{“true”}$$

$$s^5_{n,i} < -1: \quad \mathscr{N}^5_{n,i} = \text{“false.”}$$

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